A Multi-layer Extension of the Stochastic Heat Equation

by

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Contents

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Declarations

I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted for a degree at any other university

Abstract

The KPZ universality class is expected to contain a large class of random growth processes. In some of these models, there is an additional structure provided by multiple non-intersecting paths and utilisation of this additional structure has led to derivations of exact formulae for the distribution of quantities of interest. Motivated by this we study the multi-layer extension of the stochastic heat equation introduced by O'Connell and Warren in [OW11] which is the continuum analogue of the above mentioned structure. We also show that a multi-layer Cole–Hopf solution to the KPZ equation is well defined.

Chapter 1

Introduction

1.1 From Longest Increasing Subsequence to the KPZ Equation

Let $\pi \in S_n$ be a permutation of $\{1, 2, \dots, n\}$. An increasing subsequence of π is a sequence $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that $\pi(i_1) < \dots < \pi(i_k)$. For a permutation π, define $l_n(\pi)$ to be the longest length of an increasing subsequence of $\pi \in S_n$. For example the permutation

$$4\ 5\ 8\ 3\ 10\ 1\ 9\ 6\ 7\ 2$$

i.e. $\pi(1) = 4$, $\pi(2) = 5$ etc, has a longest increasing subsequence 4 5 6 7 and so $l_n(\pi) = 4$ in this case. Note that the longest increasing subsequence is not necessarily unique; 4 5 8 10 is another such sequence. A natural question is that given the uniform distribution $\mathbb{P}_n$ on S_n, what is the behaviour of the average of l_n as $n \to \infty$. This problem was posed by Ulam [Ula61] in 1961 and was subsequently known as *Ulam's problem.* Based on Monte Carlo simulations, he conjectured that the limit

$$c = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \mathbb{E}_n[l_n] \tag{1.1}$$

exists. The mathemational proof of the existence of the limit was due to Hammersley [Ham72] using subadditivity in the early 70's but he was unable to obtain the value of c. Following that Logan and Shepp [LS77] proved that $c \geq 2$ and Vershik and Kerov [VK77] independently proved that $c = 2$ and thus confirming Ulam's conjecture. It turns out that this was just the beginning of a long story and subsequently it was discovered that there is a connection with random matrix theory.

1.1.1 Young Tableaux

A partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of an integer n, denoted by $\lambda \vdash n$, is a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0$ such that $\sum_{i=1}^{l} \lambda_i = n$. One can associate to each partition λ a Young diagram which is a left justified array of boxes with λ_k boxes in the kth row. The partition λ is called the shape of the Young diagram. A Young tableau of shape $\lambda \vdash n$ is the corresponding Young diagram with integers a_{ij} placed in the (i, j)th box for each i and j. It is called *semi-standard* if the array of boxes are weakly increasing from left to right and strictly increasing from top to bottom. It is called *standard* if the integers are from the set $\{1, \ldots, n\}$ and are placed in such a way that they are strictly increasing from left to right *and* from top to bottom.

A generalised permutation of length n and row bounds (M, N) is an array of integers

$$\begin{pmatrix} i_1 & \ldots & i_n \\ j_1 & \ldots & j_n \end{pmatrix},$$

where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq M$ and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_n \leq N$ with $i_k = i_{k+1}$ implying $j_k \leq j_{k+1}$. The Robinson–Schensted–Knuth (RSK) algorithm (see for example [Ful97]) provides a one-to-one correspondence between permutations (generalised permutations) of length n and a pair of standard (semi-standard) Young tableaux (P, Q) of the same shape with n boxes. For example, the pair of standard Young tableaux corresponding to the permutation at the beginning of the introduction is

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 6 & 7 \\ \hline 3 & 5 & 9 \\ \cline{1-3} 4 & 8 \\ \cline{1-2} 10 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 7 & 9 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} 10 \\ \cline{1-1} \end{array}$$

It can be shown that for a permutation $\pi \in S_n$ the number of boxes in the top row of P (or Q) corresponding to π is equal to $l_n(\pi)$. The Plancherel measure on the set of Young diagrams with n boxes given by $\mathbb{P}^n_{\text{Planc}}[\lambda] = f_\lambda^2/n!$ is the push forward of the uniform distribution on S_n by the RSK algorithm where f_λ is the number of standard Young tableaux of shape λ. It follows by a simple conditioning that

$$\mathbb{P}_n[l_n(\pi) = l] = \frac{1}{n!} \sum_{\substack{\lambda \vdash n \\ \lambda_1 = l}} f_\lambda^2, \tag{1.2}$$

This was the starting point of the analysis of [VK77] and [LS77] who proved that $c = 2$ in the limit (1.1). In fact the description of l_n in terms of the Plancherel measure yields much more.

1.1.2 The Baik–Deift–Johansson Theorem

In 1999, a breakthrough was made by Baik, Deift and Johansson who in their celebrated paper [BDJ99a] obtained the exact asymptotic distribution of the suitably centered and scaled l_n. Their result is the following

Theorem 1.1.1 (Baik–Deift–Johansson). *Let $l_n = l_n(\pi)$ be the length of the longest increasing subsequence of $\pi \in S_n$. Under the uniform distribution on S_n, we have*

$$\lim_{n\to\infty} \mathbb{P}_n\left[\frac{l_n - 2\sqrt{n}}{n^{1/6}} \le s\right] = F_{\mathrm{GUE}}(s) \quad \textit{for all } s \in \mathbb{R},$$

where

$$F_{\mathrm{GUE}}(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)\,\mathrm{d}x\right),$$

$q(x)$ is the solution to the Painlevé II equation, $q''(x) = xq(x) + 2q^3(x)$ with $q(x) \sim \mathrm{Ai}(x)$ as $x \to \infty$.

The distribution F_{GUE} is known as the Tracy–Widom distribution. Remarkably, F_{GUE} was first shown to arise in random matrix theory by Tracy and Widom in their work [TW94] on the Gaussian Unitary Ensemble [Meh04], [AGZ10]. A GUE matrix is an $n \times n$ random Hermitian matrix with i.i.d. entries (up to symmetry) with the diagonal entries distributed as standard real Gaussian random variables and the entries above the diagonal are standard complex Gaussians. In [TW94], the authors showed that as $n \to \infty$, the distribution of the largest eigenvalue of a GUE matrix suitably centered and scaled converges to F_{GUE}.

There is an alternative expression of F_{GUE} in terms of a Fredholm determinant

$$F_{\mathrm{GUE}}(s) = \det\left(I - A|_{L^2[s,\infty)}\right) = \sum_{k=0}^\infty \frac{(-1)^k}{k!}\int_{[s,\infty)^k} \det\left[A(x_i,x_j)\right]_{i,j=1}^k \prod_{i=1}^k \mathrm{d}x_i,$$

where

$$A(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}, \tag{1.3}$$

is the Airy kernel. In the original proof of Baik–Deift–Johansson, they formulated the problem in terms of Toeplitz matrices and a Riemann–Hilbert problem. Borodin, Olshanski and Okounkov in [BOO00] provided an alternative proof by deriving a determinantal expression with a kernel involving Bessel functions for the correlation functions of the poissonised version of the Plancherel measure. Using asympototics of Bessel functions, they proved that this Bessel kernel converges to the Airy kernel and since the Plancherel measure describes the distribution of the shape of the Young diagram, via a depoissonisation argument, they were able to derive the result of Baik–Deift–Johansson. In fact they have proved much more; the joint distribution of the length of the first k rows of the Young diagram under the Plancherel measure coincides in the limit with the joint distribution of the largest k eigenvalues of a GUE matrix. This limiting distribution is the distribution of the Airy ensemble whose correlation functions are given by determinants involving the Airy kernel.

We point out here that even though one is interested in the length of the first row, it can be fruitful to consider the entire Young diagram as a whole and it can lead to exact formulae for the distribution of the quantity of interest.

1.1.3 Last Passage Percolation

In [Ham72], Hammersley related the length of the longest increasing subsequence to a Poisson process of points in the quadrant $[0,\infty)^2$. The relation is as follows: consider n distinct points (x_i, t_i), $1 \le i \le n$ in the rectangle $[0,x] \times [0,t]$. The set of points specifies a permutation π by the rule that the point with the ith smallest x-coordinate has the $\pi(i)$-th smallest t-coordinate. It can be shown that the length of the longest increasing subsequence of π (and therefore the length of the first row of the corresponding Young diagram) is equal to the maximum number of points on an up-right path from $(0,0)$ to (x,t). Now consider a Poisson process of rate 1 in $[0,\infty)$, then the number of points $N(x,t)$ in $[0,x] \times [0,t]$ is distributed as Poisson(xt) and the associated permutation of $\{1,2,\ldots,N(x,t)\}$ is uniformly distributed. Therefore, we have that

$$l_{N(x,t)} \stackrel{(d)}{=} \text{maximal number of points on an up-right path from } (0,0) \text{ to } (x,t).$$

Using this Hammersley was able to prove the existence of the limit in (1.1). In [AD95], Aldous and Diaconis associated the above set of points in the quadrant to a certain one-dimensional continuous space interacting particle system which they named *Hammersley's process*. Using this, the authors, by a hydrodynamical argument, were able to give an alternative proof of the fact that the limit c in (1.1) is equal to 2.

The up-right path model shows that the length of the longest increasing subsequence of a permutation is a special case of last passage percolation (LLP). Now consider the following variation of the model. Let $w(i,j)$, $(i,j) \in \mathbb{Z}_+^2$ be independent geometrically distributed random variables, i.e.

$$\mathbb{P}[w(i,j) = k] = (1-q)q^k, \quad k \in \mathbb{N}, \quad 0 < q < 1.$$

Define an up-right path π in $\mathbb{Z}_+^2$ from $(1,1)$ to (M,N) to be a sequence $\{(i_k,j_k)\}_{k=1}^{M+N-1}$ where each (i_k,j_k) are points in $\mathbb{Z}_+^2$ such that $(i_{k+1},j_{k+1}) - (i_k,j_k) \in \{(0,1),(1,0)\}$ and $(i_1,j_1) = (1,1)$, $(i_{M+N-1}, j_{M+N-1}) = (M,N)$. Let $\Pi_{M,N}$ be the set of all such paths. Now define

$$L(M,N) = \max_{\pi \in \Pi_{M,N}} \sum_{(i,j)\in\pi} w(i,j).$$

There is an RSK interpretation of this model. Consider an $M \times N$ matrix with independent entries $w(i,j)$ as described above. There is a one-to-one correspondence between $M \times N$ matrices with non-negative entries such that $\sum_{i=1}^M \sum_{j=1}^N w(i,j) = k$ and the set of generalised permutations of length k and row bounds (M,N). The RSK correspondence shows that there is a one-to-one mapping from such a set of generalised permutations to a pair of semi-standard Young tableaux with common shape $\lambda \vdash k$ filled respectively with integers $1, \ldots, N$ and $1, \ldots, M$. $L(M, N)$ given the constraint k is then equal to the length of the first row of the corresponding tableau. This allows one to express the distribution of $L(M, N)$ in terms of quantities associated with Young tableaux. Using combinatorial techniques and formulas involving Schur functions, Johansson proved the following in [Joh00] which was the starting point of the asymptotic analysis of $L(M, N)$. For any $M \geq N \geq 1$,

$$\mathbb{P}[L(M,N) \leq t] = \frac{1}{Z_{M,N}} \sum_{\substack{h \in \mathbb{N}^N \\ \max\{h_i\} \leq t+N-1}} \prod_{1 \leq i < j \leq N} (h_i - h_j)^2 \prod_{i=1}^{N} \binom{h_i + M - N}{h_i} q^{h_i}, \quad (1.4)$$

where $Z_{M,N}$ is the normalising constant. Compare the above formula with (1.2), both arise from conditioning on a certain set of Young tableaux, in fact one of the key inputs in the derivation (1.4) is the formula for the number of semi-standard Young tableaux of shape λ and entries in $\{1, \ldots, N\}$ obtained using Schur functions.

A key observation is the similarity between (1.4) and the distribution of the largest eigenvalue $\lambda_{\max}$ of an $N \times N$ GUE matrix,

$$\mathbb{P}[\lambda_{\max} \leq t] = \frac{1}{Z_N} \int_{(-\infty,t]^N} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{i=1}^{N} e^{-2N x_i^2} \, \mathrm{d}x_i. \quad (1.5)$$

It is well known (see [Meh04], [AGZ10]) that (1.5) can be written as a Fredholm determinant whose kernel involves the classical Hermite polynomials which as the size of the matrix tends to infinity converges to the Airy kernel. Since the convergence of the kernel implies the convergence of the Fredholm determinant itself, this gives the convergence of the centered and rescaled largest eigenvalue to the Tracy–Widom distribution.

It turns out that (1.4) can also be written as a Fredholm determinant whose kernel is the *Meixner kernel* given in terms of the Meixner polynomials. Using properties of Meixner polynomials the Meixner kernel can be analysed and subject to the appropriate scaling can be shown to converge to the Airy kernel. In summary, we have

Theorem 1.1.2 (Theorem 1.2 of [Joh00])**.** *For each $q \in (0,1)$, $\gamma \geq 1$ and $s \in \mathbb{R}$,*

$$\lim_{N \to \infty} \mathbb{P}\left[\frac{L([\gamma N], N) - \omega N}{\sigma N^{1/3}} \leq s \right] = F_{\mathrm{GUE}}(s),$$

where

$$\omega = \frac{(1 + \sqrt{q\gamma})^2}{1 - q} - 1, \quad \sigma = \left(\frac{q}{\gamma} \right)^{1/6} \frac{(\sqrt{\gamma} + \sqrt{q})^{2/3} (1 + \sqrt{q\gamma})^{2/3}}{1 - q}.$$

If instead of independent geometric random variables, the $w(i,j)$ are i.i.d. exponentially distribution i.e., $\mathbb{P}[w(i,j) \leq t] = 1 - e^{-t}$, $t \geq 0$, then a similar result to the previous theorem is true, see [Joh00] or [Har11]. In this case, the distribution of $L(M, N)$ coincides with the largest eigenvalue of a matrix from the Laguerre Unitary Ensemble (LUE). The distribution can be written as a Fredholm determinant of a kernel corresponding to Laguerre polynomials and using asymptotics for such polynomials one can show that the kernel properly scaled converges to the Airy kernel and the result follows.

Again we see here that the RSK correspondence leads one to naturally consider the entire Young diagram in the study of the length of the first row which leads to an exact formula for its distribution from which one can take asymptotics.

1.1.4 Finite Temperature Discrete Directed Polymer

The models above are the zero temperature limit of discrete polymers in random media. In the finite temperature setting, instead of maximising over all possible paths, one takes the sum over all paths. More precisely, define

$$Z^{\beta}(n,x) = \sum_{\pi \in \Pi_{0,x;n}} \exp\left(\beta \sum_{i=0}^{n} w(i,\pi(i))\right),$$

where $\Pi_{0,x;n}$ is the collection of simple symmetric random walk trajectories from 0 at time 0 to x at time n and $w(i,j)$ are i.i.d. random variables. The parameter β is known as the inverse temperature. Z^{β} is the partition function of the discrete directed polymer and its logarithm is called the free energy. Sending $\beta \to \infty$ in $\beta^{-1} \log Z^{\beta}$ we recover the formula for $L(M,N)$.

An interesting example that is solvable is the log-gamma polymer introduced by Seppäläinen and further studied by Corwin et al in [COSZ14]. Crucial in their work is an extension of the RSK correspondence called the geometric RSK. The algorithm takes as input an $n \times N$ matrix with strictly positive real entries and outputs a triangular array $\tau = (\tau_{jk} : 1 \leq k \leq j \leq N)$ of positive real numbers. Let $\mathbb{T}_N$ be the set of all such arrays. The algorithm can be described in terms of a sequence of row insertions where each insertion is a procedure for taking a vector of N elements and a triangular array $\tau \in \mathbb{T}_N$ as input and outputing a new array $\tau' \in \mathbb{T}_N$. Then, given an input matrix and an initial array, one obtains a new array by successively inserting each row of the input matrix into the initial array. The output array is the analogue of the P-tableau in the RSK correspondence and the analogue of the shape of the tableau is the bottom row of the array.

Now fix $N \geq 1$ and consider a semi-infinite matrix $d = (d_{ij} : i \geq 1, 1 \leq j \leq N)$ with d_{ij} positive, then an evolution of an array $\big(\tau(n) \in \mathbb{T}_N : n = 0,1,2,\ldots\big)$ is defined by successive insertions of rows of d into the initial array $\tau(0) \in \mathbb{T}_N$. An interesting case is when the initial array is empty. In this case there is a relation between the process $\{\tau(n)\}_{n\geq 0}$ and non-intersecting lattice paths. Let $d^{[1,n]} = (d_{ij} : 1 \leq i \leq n, 1 \leq j \leq N)$ be the first n rows of d. For $1 \leq k \leq j \leq N$, let $\Pi^{k}_{n,j}$ denote the set of k-tuples $\pi = (\pi_1,\ldots,\pi_k)$ of non-intersecting up-right paths in $\mathbb{Z}^2$ such that for each $1 \leq r \leq k$, π_r is an up-right path from $(1, r)$ to $(n, j + r - k)$. For $1 \le k \le j \le N$, define

$$z_k(n, j) = \sum_{\pi \in \Pi^k_{n,j}} wt(\pi), \quad wt(\pi) = \prod_{r=1}^{k} \prod_{(i,j) \in \pi_r} d_{ij}. \tag{1.6}$$

Define an array $\tau(n) = (\tau_{jk}(n) : 1 \le j \le N, 1 \le k \le j \wedge n)$ recursively by

$$\tau_{j1}(n) \cdots \tau_{jk}(n) = z_k(n, j). \tag{1.7}$$

It turns out that $\tau(n)$ defined in this way is the same as the result of row insertions of d into an empty array. Clearly, $z_k(n, j)$ can be written as $\sum_{\pi \in \Pi^k_{n,j}} \exp\left(\sum_{r=1}^{k} \sum_{(i,j)\in\pi_r} \log d_{ij}\right)$ and so $z_k(n, j)$ can be interpreted as a multi-layer partition function of a directed polymer. The aforementioned log-gamma polymer partition function is $z_1(n, N) = \tau_{N1}(n)$ with the entries d_{ij} of the input matrix being independent inverse-gamma distributed random variables ($d_{ij} \sim \Gamma^{-1}(\theta_{ij})$), that is $\mathbb{P}[d_{ij} \in \mathrm{d}x] = \Gamma(\theta_{ij})^{-1} x^{-\theta_{ij}-1} e^{-1/x}\, \mathrm{d}x$ where $\theta_{ij} > 0$ is a parameter. The geometric RSK correspondence applied to this input matrix d results in a Markov chain $(\tau(n), n \ge 0)$ with state space $\mathbb{T}_N$ whose transition kernel can be explicitly computed under certain constraints on the parameter θ_{ij}. Using the theory of Markov functions [RP81] and a certain intertwining relation, the authors also showed that the bottom row of the array also has a Markovian evolution. One of the main results in [COSZ14] is an N-fold integral formula for the Laplace transform of the polymer partition function $z_1(n, N)$. Using this formula, Borodin, Corwin and Remenik in [BCR13] were able to derive a Fredholm determinant expression for the Laplace transform of $z_1(n, N)$ which lends itself to asymptotic analysis. Their result is that the limiting distribution under $n^{1/3}$ scaling of the log-gamma polymer free energy is the GUE Tracy–Widom distribution. More precisely,

Theorem 1.1.3 (Theorem 1 of [BCR13])**.** *There exists $\gamma^* > 0$ such that for any $\gamma \in (0, \gamma^*)$ if $d_{ij} \sim \Gamma^{-1}(\gamma)$ for all i, j, we have*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\log z_1(n, n) - n\mu_\gamma}{n^{1/3}} \le s\right] = F_{\mathrm{GUE}}\left(\left(\frac{\sigma_\gamma}{2}\right)^{-1/3} s\right)$$

for some explicit μ_γ and σ_γ.

1.1.5 The Kardar–Parisi–Zhang Equation

A way to obtain a continnum version of Z^β is to replace the collection of i.i.d. random variables $w(i, j)$ with space-time white noise and the random walk bridge by a Brownian bridge. Such an object is the partition function of the continuum directed random polymer (CDRP) [AKQ14a], [Cor12]. More precisely, the partition function is given by

$$Z(t, x) = \mathbb{E}^b_{0,x;t}\left[\mathscr{E}\mathrm{xp}\left(\int_0^t \dot{W}(s, b(s))\, \mathrm{d}s\right)\right], \tag{1.8}$$

where b is a Brownian bridge starting at 0 at time 0 and ending at x at time t, $\mathbb{E}^b_{0,x;t}$ is the corresponding expectation and $\dot{W}$ is space-time white noise. Formally,

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(x-y)\delta(t-s).$$

$\mathscr{E}\mathrm{xp}$ is the Wick exponential defined by $\mathscr{E}\mathrm{xp}(M_t) := \exp(M_t - \frac{1}{2}\langle M, M\rangle_t)$ for a martingale $(M_t)_{t\geq 0}$. The expression (1.8) is not well defined as it is unclear how one would define the integral of the white noise along the Brownian path and also to exponentiate such an expression since space-time white noise is not a function but rather a generalised function. There are a number of ways to make sense of (1.8). One is to interpret it as a short hand for the chaos expansion:

$$Z(t,x) = \sum_{k=0}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} \frac{p_{t-s_k}(x-y_k)p_{s_k-s_{k-1}}(y_k-y_{k-1})\dots p_{s_1}(y_1)}{p_t(x)} \, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}), \quad (1.9)$$

where $\Delta_k(t) := \{0 < s_1 < \dots < s_k < t\}$ and $p_t(x-y) = (2\pi t)^{-1/2}e^{-(x-y)^2/2t}$. The integral is a multiple stochastic integral in the sense of Walsh see Appendix A. Observe that $u(t,x) := p_t(x)Z(t,x)$ is the solution to the stochastic heat equation (SHE):

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) + u(t,x)\dot{W}(t,x), & t>0, x\in\mathbb{R}, \\ u(0,x) = \delta_0(x), & x\in\mathbb{R}. \end{cases} \quad (1.10)$$

By a solution to the above we mean a random field u satisfying the mild form of the equation:

$$u(t,x) = p_t(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)u(s,y)\, W(\mathrm{d}s, \mathrm{d}y). \quad (1.11)$$

Notice that by iterating the above multiple times we obtain the chaos expansion $p_t(x)Z(t,x)$.

A different approach to make sense of (1.8) is to considered a mollified version W^ε of the white-noise. For such a noise, the expression (1.8) is now rigorious and moreover it solves a certain stochastic partial differential equation driven by the noise W^ε. Bertini–Cancrini [BC95] showed that as one takes away the smoothing the expression converges to the solution to the stochastic heat equation (1.10).

It is well known that the logarithm of the solution to the SHE (1.11) is the Cole–Hopf solution [BG97] to the Kardar–Parisi–Zhang equation (KPZ) [KPZ86] with narrow wedge initial data (for a survey see [Cor12]):

$$\partial_t h(t,x) = \frac{1}{2}\Delta_x h(t,x) + \frac{1}{2}\big(\partial_x h(t,x)\big)^2 + \dot{W}(t,x). \quad (1.12)$$

The equation was introduced in the 1986 paper [KPZ86] of Kardar, Parisi and Zhang in the study of randomly growing interfaces and since then the equation has gathered much interest from both mathematicians and physicists. Intuitively, the equation says that the change in time of the growth interface is due to a smoothing effect represented by the term

$\Delta_x h$, rotationally invariant, slope dependent growth represented by $(\partial_x h)^2$ and space-time independent random forcing modelled by $\dot{W}$. It is believed that discrete models which share the above three key features should display similar limiting (e.g. in time or system size) fluctuations (which may differ depending on the initial condition) and have the same scaling exponent irrespective of the details of the mechanics of the growth model. In other words they form a universality class which is now commonly called the *KPZ universality class*. Much work have been done over the years since the paper [KPZ86] either by numerical methods or mathematically non-rigorous approches to determine these universal scaling exponents and limiting distributions. Then a breakthrough was made in the late 1990s by Baik, Deift and Johansson who found the limiting distribution and the characteristic scaling exponent of a last passage percolation (LLP) model (see the above discussion) which was predicted to be in the KPZ universality class, note that LLP is related to the corner growth model. It is now believed that the 1/3 scaling is universal among all models in the universality class and there is a further subclass depending on the initial condition where the GUE Tracy–Widom distribution describes the limiting fluctuations. See [Cor12] for a description of the other subclasses.

The distribution of the solution to the KPZ equation remained unknown until 2010 when the authors of [ACQ11] and [SS10] independently obtained the exact formula for the distribution of the solution to the KPZ equation which allowed the derivation of its long time asymptotics and hence showed that the solution of the KPZ equation displays the appropriate limiting behaviour and scaling associated with the KPZ universality class.

Theorem 1.1.4. *Let $h(t,x)$ be the Cole–Hopf solution to the KPZ equation with narrow wedge initial data, then*

$$\lim_{t\to\infty} \mathbb{P}\Big[h(t,x) - \frac{x^2}{2t} - \frac{t}{24} \geq -2^{-1/3}t^{1/3}s\Big] = F_{\mathrm{GUE}}(s).$$

By the Feynman–Kac formula, the Cole–Hopf solution to the KPZ equation is, up to a deterministic translation, the free energy of the continuum directed random polymer. With this interpretation, Theorems 1.1.1, 1.1.2, 1.1.3 and 1.1.4 suggest that the Cole–Hopf solution to the KPZ equation can be regarded as the continuum and finite temperature analogue of the longest increasing subsequence of a random permutation.

1.2 A Multi-layer Extension of the Stochastic Heat Equation

We have mentioned that a similar result to Theorem 1.1.1 also holds for the other eigenvalues of a GUE matrix. The fact that the distribution of the length of the second row of the Young tableau, obtained from applying the RSK correspondence to a random permutation, converges to the limiting distribution of the second largest eigenvalue of a GUE matrix was proved by Baik, Deift and Johansson in [BDJ99b]. The connection between the first k row lengths and the k largest eigenvalues was proved independently by Borodin–Okounkov–Olshanski [BOO00], Johansson [Joh01a] and Okounkov [Oko00].

There is an interpretation due to Greene [Gre74] of the length of these other rows of a Young diagram with permutations. For a permutation π, a k-increasing subsequence is a union of k disjoint increasing subsequences in π. Define a sequence of integers $\lambda_1, \lambda_2, \ldots$ by letting $\lambda_1 + \cdots + \lambda_k$ be the length of the longest k-increasing subsequence of π. Then $(\lambda_1, \lambda_2, \ldots)$ is simply the common shape of the pair of the Young tableaux corresponding to π. For example, for the permutation specified in the beginning of the introduction, we have $\lambda_1 = 4$, a longest 2-increasing subsequence is the union of 4 5 8 9 and 3 6 7 and so $\lambda_1 + \lambda_2 = 7$, a longest 3-increasing subsequence is the union of 4 5 8 9, 3 6 7 and 1 2, so that $\lambda_1 + \lambda_2 + \lambda_3 = 9$. There's only one element remaining so $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 10$. Thus, the shape of the corresponding Young tableau is (4,3,2,1).

Another way to think of this is in terms of non-intersecting up-right paths on the lattice. Let $\Pi^k_{M,N}$ be the set of k-tuples $\pi = (\pi_1, \ldots, \pi_k)$ of non-intersecting up-right paths in $\mathbb{Z}^2_+$ where each π_r is a path from $(1, r)$ to $(M, N + r - k)$. Now define for $k \geq 2$

$$L_k(M,N) = \max_{\pi \in \Pi^k_{M,N}} \sum_{r=1}^{k} \sum_{(i,j) \in \pi_r} w(i,j), \tag{1.13}$$

where $w(i, j)$ are as in Section 1.1.3. Then

$$L_k(M,N) = \lambda_1 + \cdots + \lambda_k. \tag{1.14}$$

Compare (1.13) and (1.14) with equations (1.6) and (1.7) where the quantities involved are also defined in terms of multiple non-intersecting paths on a lattice.

We have seen that the Cole–Hopf solution to the KPZ is the continuum analogue of the length of the first row of a Young tableau, so a natural question to ask is what is the analogue of the second row, the third row and so on in the KPZ setting. In [OW11], O'Connell and Warren introduced the following: for x, $y \in \mathbb{R}$, $n = 1, 2, \ldots$ define

$$Z_n(t,x,y) = p_t(x-y)^n \left(1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s}, \mathbf{y}; t, x, y)\, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) \right), \tag{1.15}$$

where $\mathbf{s} = (s_1, \ldots, s_k)$, $\mathbf{y} = (y_1, \ldots, y_k)$, R_k is the k-point correlation function for a collection of n non-intersecting Brownian bridges which all start at x at time 0 and end at y at time t.

Observe that $Z_1(t, 0, x) = u(t, x)$ is the solution to the SHE (1.10). Just as (1.8) can be considered as the short hand for (1.9), the short hand for the chaos expansion (1.15) above is the following Feynman–Kac formula

$$Z_n(t,x,y) = p_t(x-y)^n \mathbb{E}^X_{x,y;t}\left[\mathscr{E}\mathrm{xp}\left(\sum_{i=1}^n \int_0^t \dot{W}(s,X^i_s)\,\mathrm{d}s\right)\right],$$

where $(X^1_s,\dots,X^n_s, 0\le s\le t)$ denote the trajectories of n non-intersecting Brownian bridges which all start at x at time 0 and all end at y at time t and the expectation is with respect to such Brownian bridges. So on one hand, $Z_n(t,x,y)$ is a multi-layer extension to the solution to the stochastic heat equation given by the chaos expansion (1.9). On the other hand, Z_n can be considered as the multi-layer extension to the partition function of the continuum directed random polymer. It should not be difficult to make sense of the above Feynman–Kac formula by smoothing the white noise as in the case of the SHE.

In each of the discrete models described above, there is a multi-layer structure provided either by multiple non-intersecting up-right paths on lattices or the entire Young diagram via the RSK correspondence and the work in the above mentioned references have shown that in some cases, utilisation of this multi-layer structure have led to derivations of exact formulae for the distribution of quantities of interest. The above mentioned discrete models provide examples of what is called *integrability* or *exact solvability*. The motivation for introducing the partition functions Z_n, which is the continuum analogue of the multi-layer structure mentioned above, is that they may provide insight to integrability in the continuum setting.

Motivated by (1.6) and (1.14), we define the nth layer of the solution to the KPZ equation which is a multi-layer extension of the free energy of the CDRP by

$$h_n(t,x) = \log\left(\frac{Z_n(t,0,x)}{Z_{n-1}(t,0,x)}\right), \quad Z_0 \equiv 1. \tag{1.16}$$

Since the Cole–Hopf solution to the KPZ is the analogue of the length of the top row of a Young diagram, $\big(h_n(t,x) : n = 2,3,\dots\big)$ can be seen as the analogue of the rest of the row lengths of the Young diagram in the KPZ setting. A natural question to ask is whether an extension of Theorem 1.1.4 holds for the additional layers h_n. Since Dyson's Brownian motion which is the time evolving version of the eigenvalues of a GUE matrix has the multi-line Airy process, introduced in [PS02] as its scaling limit, it is reasonable to believe that $\big(h_n(t,x) : x\in\mathbb{R}, n = 1,2,\dots\big)$ should with appropriate scaling converge to the multi-line Airy process as $t\to\infty$. This is also supported by the one-point convergence result Theorem 1.1.4 and the fact that the Airy process which is the top line of the multi-line Airy process has one-point distribution equal to the GUE Tracy–Widom distribution. More recently, Nguyen and Zygouras in [NZ15] obtained formulae for the joint Laplace transform of the log-gamma polymer partition function at different space-time points by establishing variants of the geometric RSK correspondence. Using these formulae, they were able to show formally the convergence of the joint distribution of two partition functions at equal time to the two-point function of the Airy process. So far this is currently out of reach in the continuum setting even for the first layer, the Cole–Hopf solution to the KPZ equation. Nevertheless, it is interesting to derive properties of $(Z_n, n = 1, 2, \ldots)$ as it may further our understanding of the KPZ equation.

From the definition (1.15), it is not clear whether $h_n(t, x)$ in (1.16) is well-defined because that would require Z_n to be everywhere strictly positive almost surely. We will prove that this can in fact be done. More precisely, we will prove that there exists a version of Z_n such that $(t, x, y) \mapsto Z_n(t, x, y)$ is continuous and almost surely for all $t > 0$ and x, $y \in \mathbb{R}$, $Z_n(t, x, y) > 0$. Moreover, this regularity of Z_n can be further used to show that for all $n \geq 1$ the *multi-layer process*

$$\big(Z_1(t, x, \cdot), \ldots, Z_n(t, x, \cdot), t \geq 0\big), \tag{1.17}$$

is a Markov process with state space $C(\mathbb{R}) \times \cdots \times C(\mathbb{R})$ which is the main contribution of this thesis. This result can be considered as the continuum analogue of the fact that the evolution of the triangular array $(\tau(n), n \geq 0)$ obtained from applying the geometric RSK correspondence to an input matrix discussed above is a Markov chain. Moreover, it was shown in [OW11, Proposition 3.3 and 3.7] by considering a smooth space-time potential that $(Z_n, n \geq 1)$ should satisfy a system of coupled SPDEs, however unfortunately it is not immediately obvious that such SPDEs make sense in the white noise setting. Nevertheless, it does suggests that the process should have a Markovian evolution.

In order to prove the Markov property of (1.17), we consider the following: for $n \geq 1$, $t > 0$ and $\mathbf{x}$, $\mathbf{y} \in W_n := \{\mathbf{y} \in \mathbb{R}^n : y_1 \geq y_2 \geq \cdots \geq y_n\}$, define

$$K_n(t, \mathbf{x}, \mathbf{y}) = p_n^*(t, \mathbf{x}, \mathbf{y}) \Bigg(1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s}, \mathbf{y}'; t, \mathbf{x}, \mathbf{y})\, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}')\Bigg),$$

and for $\mathbf{x}$, $\mathbf{y} \in W_n^\circ$, let

$$M_n(t, \mathbf{x}, \mathbf{y}) = \frac{K_n(t, \mathbf{x}, \mathbf{y})}{\Delta(\mathbf{x})\Delta(\mathbf{y})},$$

where $p_n^*(t, \mathbf{x}, \mathbf{y}) = \det[p_t(x_i - y_j)]_{i,j=1}^n$ and $R_k(\mathbf{s}, \mathbf{y}'; t, \mathbf{x}, \mathbf{y})$ is the k-point correlation of a collection of n non-intersecting Brownian bridges starting at $\mathbf{x}$ at time 0 and ending at $\mathbf{y}$ at time t. $\Delta(\mathbf{x}) = \prod_{1 \leq i < j \leq n}(x_i - x_j)$ is the Vandermonde determinant. A conjecture in [OW11] is that the following integral formula which relates for each n, $\big(Z_1(t, x, \cdot), \ldots, Z_n(t, x, \cdot)\big)$ to $M_n(t, x\mathbf{1}, \cdot)$, $\mathbf{1} = (1, \ldots, 1)$ holds

$$M_n(t, x\mathbf{1}, \mathbf{y}) = \frac{1}{\Delta(y)} \prod_{i=1}^n u(t, x, y_i) \int_{\mathrm{GT}(\mathbf{y})} \prod_{k=1}^{n-1} \prod_{i=1}^{n-k} \frac{1}{t} \frac{Z_{k-1}(t, x, y_i^{n-k}) Z_{k+1}(t, x, y_i^{n-k})}{Z_k(t, x, y_i^{n-k})^2}\, \mathrm{d}y_i^{n-k}, \tag{1.18}$$

where $u(t, x, y)$ is the solution to the stochastic heat equation with initial data δ_x each driven by the same white noise and $\mathrm{GT}(\mathbf{y})$ is the Gelfand–Tsetlin polytope:

$$\mathrm{GT}(\mathbf{y}) := \{(\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^{n-1}) \in W_1 \times W_2 \times \cdots \times W_{n-1} : \mathbf{y}^1 \prec \mathbf{y}^2 \prec \cdots \prec \mathbf{y}^{n-1} \prec \mathbf{y}\},$$

where for $\mathbf{z} \in W_{n-1}$ and $\mathbf{y} \in W_n$, we write $\mathbf{z} \prec \mathbf{y}$ if $y_1 \geq z_1 > y_2 \geq \ldots > y_{n-1} \geq z_{n-1} > y_n$. This formula together with the Markov property of $M_n(t, \mathbf{x}, \cdot)$, which follows from [OW11, Corollary 6.2], would imply the Markov property $\big(Z_1(t, x, \cdot), \ldots, Z_n(t, x, \cdot), t \geq 0\big)$. The integral formula was conjectured and proved only in the case $n = 2$ in [OW11], the obstacle being that the continuity of M_n on the whole of the Weyl chamber was only established in the $L^2(\dot{W})$ sense and a proof of its strict positivity was unavailable. In Chapter 3, we will prove that M_n has a version that is jointly continuous over $(0, \infty) \times W_n \times W_n$ and in particular when all the coordinates of $\mathbf{x}$ coincide and likewise for $\mathbf{y}$, M_n agrees with Z_n upto a multiplicative constant:

$$M_n(t, a\mathbf{1}, b\mathbf{1}) = c_n t^{-n(n-1)/2} Z_n(t, a, b), \quad c_n = \left(\prod_{i=1}^{n-1} i! \right)^{-1}$$

Thus, M_n can be thought of as an extension of Z_n from the boundary of the Weyl chamber to its interior. In Chapter 4, we prove that $M_n(t, x, y) > 0$ for all $t > 0$, x, $y \in W_n$ almost surely. The strict positivity and continuity of Z_n then follows immediately.

It was shown in [OW11, Proposition 3.2] that M_n has the determinantal expression

$$M_n(t, \mathbf{x}, \mathbf{y}) = \frac{\det[u(t, x_i, y_j)]_{i,j=1}^n}{\Delta(\mathbf{x})\Delta(\mathbf{y})},$$

where $u(t, x, y)$ is the solution to the SHE with initial data δ_x and since u is continuous in its spatial variables it is clear that M_n is continuous in the interior $W_n^\circ \times W_n^\circ$. Since $p_n^*(t, \mathbf{x}, \mathbf{y})/\Delta(\mathbf{x})\Delta(\mathbf{y})$ is a smooth function of $(\mathbf{x}, \mathbf{y})$ over $\mathbb{R}^n \times \mathbb{R}^n$ and since the k-point correlation function R_k extends continuously to the boundary of the Weyl chamber, see Section 3.2.1, we see from its chaos expansion that $M_n(t, \mathbf{x}, \mathbf{y})$ is defined for $\mathbf{x}$, $\mathbf{y} \in \partial W_n$. The issue is its continuity at the boundary of the Weyl chamber. If $(x, y) \mapsto u(t, x, y)$ is a smooth function then as $\mathbf{x} \to x\mathbf{1}$ and $\mathbf{y} \to y\mathbf{1}$, $M_n(t, \mathbf{x}, \mathbf{y})$ converges to a limit that is proportional to $Z_n(t, x, y)$ given by (1.20) below. To see this, define the difference operator $\delta^0 f(x) = f(x)$, $\delta f(x) = f(x+h) - f(x)$ and $\delta^n f(x) = \delta\big(\delta^{n-1} f(x)\big)$ for a function $f : \mathbb{R} \to \mathbb{R}$ and $h > 0$. Writing $x_i = x + (n-i)h$ and $y_i = y + (n-i)k$, then by [Chu09, Lemma 14].

$$M_n(t, \mathbf{x}, \mathbf{y}) = c_n^2 \frac{\det[\delta_x^{i-1} \delta_y^{j-1} u(t, x, y)]_{i,j=1}^n}{h^{n(n-1)/2} k^{n(n-1)/2}},$$

where δ_x, δ_y is the difference operator in the x and y variable respectively. Taking the limit as $h, k \to 0$, we obtain, up to a constant depending on n and t, the right hand side of (1.20).

However, we know that the solution to the stochastic heat equation is only Hölder continuous of order up to 1/2 and so the continuity of $M_n(t, \mathbf{x}, \mathbf{y})$ at the boundary of the Weyl chamber is not immediately obvious. For example, in the case $n = 2$, consider the function $M_2(\mathbf{y}) := \det[g_i(y_j)]/(y_1 - y_2)$ where $g_1(y) = |y|$ and $g_2(y) \equiv 1$ then in the limit as $y_1, y_2 \to y$, $M_2(\mathbf{y})$ converges to a limiting function that is equal to 1 for $y > 0$, -1 for $y < 0$ with a discontinuity at 0. Furthermore, if one takes as g_i independent Brownian motions then there is no limiting function at the boundary at all.

The key to proving the regularity of M_n is the fact that it satisfies a certain SPDE. We will show in Chapter 3 that $M_n(t, \mathbf{x}, \mathbf{y})$ satisfies the following integral equation

$$M_n(t, \mathbf{x}, \mathbf{y}) = \frac{p_n^*(t, \mathbf{x}, \mathbf{y})}{\Delta(\mathbf{x})\Delta(\mathbf{y})} + A_n \int_0^t \int_{\mathbb{R}^n} Q_{t-s}(\mathbf{y}, \mathbf{y}') M_n(s, \mathbf{x}, \mathbf{y}') \, \mathrm{d}y'_* \, W(\mathrm{d}s, \mathrm{d}y'_1), \tag{1.19}$$

where $A_n = n/(n-1)!$, $\mathrm{d}y'_* = \mathrm{d}y_2 \ldots \mathrm{d}y_n$ and $Q_t(\mathbf{y}, \mathbf{y}') = \Delta(\mathbf{y})^{-1} p_n^*(t, \mathbf{y}, \mathbf{y}')\Delta(\mathbf{y}')$ is the transition density of Dyson's Brownian motion. This can be seen by substituting the chaos expansion of M_n into the above equation and using the definition (see Section 3.2.1) of the k-point correlation function R_k. Comparing the above equation with the mild form of the SHE (1.11), we see that they are of a similar form and thus we can consider equation (1.19) as the mild form of a multi-dimensional SHE. It is worth pointing out that (1.19) is not the conventional definition of a multi-dimensional SHE which would involve a higher dimensional noise and a single multi-dimensional Brownian path. However, for dimensions greater than one, such an equation driven by space-time white noise does not have a solution that is a function. There is no such issue with equation (1.19) and this allows us to use SPDE techniques to prove the continuity and strict positivity of M_n. Moreover, it is now natural that $(M_n(t, \mathbf{x}, \cdot), t \geq 0)$ has the Markov property since it satisfies an evolution equation. In the next section, we shall recall some existing results in the literature on the one-dimensional SHE.

It turns out that Z_n has connections with classical integrable systems. It was shown in [OW11] that in the case with a smooth space-time potential in place of the space-time white noise, Z_n is given by the bi-directional Wronskian

$$Z_n(t, x, y) = c_n t^{n(n-1)/2} \det[\partial_x^{i-1} \partial_y^{j-1} u(t, x, y)]_{i,j=1}^n, \tag{1.20}$$

where $u(t, x, y)$ is the solution to the heat equation with initial data δ_x driven by the smooth potential.

Let $\tau_n = \det[\partial_x^{i-1} \partial_y^{j-1} u(t, x, y)]_{i,j=1}^n$ then τ_n satisfy the two-dimensional Toda equations (2DTE)

$$\partial_{xy} \log \tau_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2},$$

with the convention that $\tau_0 \equiv 1$. This can be seen by evaluating the above derivative and comparing the result with the Jacobi identity for determinants [Hir04, equation 2.73], see Section 5.1 for more details. In [OW11], the authors by deriving identities involving higher derivatives of τ_n showed that the integral formula (1.18) for general n holds in the case of a smooth space-time potential. In the white noise setting, none of these derivatives exist but nevertheless we can still show, using the continuity and strict positivity of M_n, that (1.18) holds and moreover for each fixed time t, the process $\tilde{Z}_n := c_n^{-1} Z_n$ satisfies an integrated form of the 2DTE. By this we mean that for any $x_1 > x_2$ and $y_1 > y_2$

$$\log \frac{\tilde{Z}_n(t, x_1, y_1)}{\tilde{Z}_n(t, x_1, y_2)} - \log \frac{\tilde{Z}_n(t, x_2, y_1)}{\tilde{Z}_n(t, x_2, y_2)} = t^{-n(n-1)/2} \int_{x_2}^{x_1} \int_{y_2}^{y_1} \frac{\tilde{Z}_{n-1}(t, x, y)\tilde{Z}_{n+1}(t, x, y)}{\tilde{Z}_n(t, x, y)^2} \, \mathrm{d}y\mathrm{d}x.$$

This suggests that one can interpret the fixed time Cole–Hopf solution to the KPZ equation with narrow wedge initial condition as the first element of the two-dimensional Toda chain.

We mention briefly here the work of Corwin and Hammond. In [CH15], the authors constructed an $\mathbb{N}$-indexed ensemble $(\mathcal{H}_n^t : n \in \mathbb{N})$ of random continuous curves $\mathcal{H}_n^t : \mathbb{R} \to \mathbb{R}$, which they named the KPZ line ensemble where the lowest indexed curve $\mathcal{H}_1^t$ is equal in distribution to the Cole–Hopf solution $h(t, \cdot)$ to the KPZ equation with narrow wedge initial data. It is expected but not yet proved that their construction is equal to $(h_n(t, \cdot) : n \in \mathbb{N})$ defined in (1.16), see [CH15, Conjecture 2.18].

1.3 A Review of the One-dimensional Stochastic Heat Equation

Equation (1.19) has a similar structure to the mild form of the multiplicative stochastic heat equation (also called the parabolic Anderson model (PAM) for the particular case discussed above) and so it is susceptible to analysis using techniques in the SPDE literature. For an introduction, see [Wal86], [Kho09] or [Kho14].

We are mainly interested in the continuity and the strict positivity of the solution to (1.19) and its one-dimensional counterpart (1.23) defined below. The continuity of the solution SHE has been well studied. In [Wal86] for bounded initial data, the solution to the SHE in a bounded spatial domain has been shown to be Hölder continuous with indices up to 1/2 in space and up to 1/4 in time. For initial data μ being a positive Borel measure on $\mathbb{R}$ satisfying

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \sqrt{t}\big(\mu * p_t\big)(x) < \infty, \quad \text{for all } T > 0, \tag{1.21}$$

Bertini and Cancrini in [BC95] claimed that the solution to the PAM is Hölder with the same indices as above. Note that the class of initial data considered in [BC95] includes a delta type initial data which is the initial data of interest in this thesis.

For a measure with compact support as initial data, Conus et al in [CJKS14] proved that the SHE with the Laplacian replaced with the infinitesimal generator of a symmetric Lévy process is Hölder continuous in space with indices up to 1/2. However, they did not prove the time continuity of the solution. For a signed Borel measure over $\mathbb{R}$ such that

$$\big(|\mu| * p_t\big)(x) < \infty, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}, \tag{1.22}$$

Chen and Dalang proved in [CD14] that the solution to a nonlinear SHE is Hölder continuous with indices up to 1/2 in space and 1/4 in time. The class of initial data (1.22) includes a delta type initial data and permits certain exponential growth at infinity, for example

$\mu(x) = f(x)\,\mathrm{d}x = e^{a|x|^p}$, $a > 0$, $p \in (0,2)$. For results on different variations of the SHE see [SSS02], [DD05], [CD15b] and the references therein.

In each case the tool used to prove the continuity of the solution is Kolmogorov's continuity criterion. Denote the stochastic integral term of (1.11) by $I(t,x)$ then the key is to show that

$$\mathbb{E}[|I(t,x) - I(t',x')|^p] \leq C\big(|x-x'|^{p/2} + |t-t'|^{p/4}\big),$$

for p large enough. This in turn requires showing some continuity estimate for the heat kernel which gets increasingly involved for increasingly less regular initial data due to the pth moments $\mathbb{E}[|u(t,x)|^p]$ of the solution being unbounded as $t \downarrow 0$ or as $x \to \infty$ or both. However for certain initial data such as a delta function, even though the pth moments blow up as time $t \downarrow 0$, they are for any fixed positive times uniformly bounded in space and thus one can in effect isolate the effects of the initial data by solving the equation for a small time and then start afresh with the current solution as the new initial value. The important point is that for initial data that is uniformly bounded in space, the continuity of the corresponding solution is much easier to obtain. This was the approach taken in [CJKS14] and the approach we will take for M_n as it fits into the situation described above.

The strict positivity of the solution to the stochastic heat equation was first proved by Mueller in [Mue91], in fact he proved a strong comparison principle of which the strict positivity is a corollary. He proved that if the initial data f is non-negative, continuous with compact support with $f(x) > 0$ for some x, then for all $t > 0$

$$\mathbb{P}[u(t,x) > 0 \text{ for every } x \in \mathbb{R}] = 1.$$

Bertini and Cancrini proved a weak comparison principle using the Feynman–Kac formula and used it to extend Mueller's result to initial data satisfying (1.21). Shiga in [Shi94] proved the stronger statement

$$\mathbb{P}[u(t,x) > 0 \text{ for every } x \in \mathbb{R} \text{ and every } t > 0] = 1,$$

for initial data being a continuous function such that the tails grow no faster than $e^{\lambda|x|}$ for all $\lambda > 0$. More recently, Moreno Flores in [Flo14] proved the strict positivity of the solution for delta initial conditions, using a convergence result of a discrete polymer model to the SHE, see [AKQ14b]. Chen and Kim [CK14] further generalised the strict positivity result to the fractional SHE for measure-valued initial data satisfying (1.22) by adapting Shiga's method.

In all of the proofs above (except for the polymer proof) a key result is a large deviation estimate on the stochastic integral term of the solution. Mueller proved such a result using the fact that integrals of the type $\int_0^t \int_{\mathbb{R}} f(s,y)\ W(\mathrm{d}s, \mathrm{d}y)$ can be considered as a time-changed Brownian motion. Chen and Kim using a method of [CJK12] derived a similar estimate for the fractional SHE using Kolmogorov's continuity criterion. We will follow the approach of [CK14] since we will first derive the necessary estimates to apply the continuity criterion to prove Hölder continuity anyway.

1.4 Outline of the Thesis

The outline of the thesis is as follows. In Chapter 2, we study the equation

$$v(t,x,y) = \frac{p_t^*(x,y)}{xy} + \int_0^t \int_0^\infty q_{t-s}(y,z) v(s,x,z)\, W(\mathrm{d}s, \mathrm{d}z), \tag{1.23}$$

where $p_t^*(x,y) = p_t(x-y) - p_t(x+y)$ and $q_t(x,y) = x^{-1} p_t^*(x,y) y$ is the transition density of a three-dimensional Bessel process BES(3) process. This Bessel process can be realised as the eigenvalues of a traceless Hermitian matrix with independent standard Brownian motions as its entries. On the other hand, it is the Doob h-transform of Brownian motion killed on the half-line. With this in mind, the BES(3) process can be considered as the one dimensional analogue of Dyson Brownian motion. Therefore, equation (1.23) is the natural one-dimensional analogue to (1.19). We will show that there exists a unique solution to the above equation and moreover the solution has a version that is jointly continuous in (t,x,y) over $(0,\infty) \times [0,\infty) \times [0,\infty)$. An immediate corollary of the spatial continuity of v is that the solution to the stochastic heat equation on the half-line $[0,\infty)$ with Dirichlet boundary condition at 0 has a derivative at 0.

In Chapter 3, we study (1.19) and prove that it has a unique solution that is Hölder continuous over $(0,\infty) \times W_n \times W_n$ with the familiar indices of 1/2 and 1/4. We also provide upper bounds on the pth moments of $M_n(t,\mathbf{x},\mathbf{y})$ in terms of local times of non-intersecting Brownian bridges. In Chapter 4, we prove a strong comparison principle for equation (1.19) which would imply the strict positivity of $M_n(t,\mathbf{x},\mathbf{y})$. Finally, the integral formula (1.18) and the Markov property of the multi-layer process (1.17) will be proved in Chapter 5.

Chapter 2

A One Dimensional Case

2.1 Introduction

We study the following integral equation:

$$\begin{aligned} v(t,y) &= \int_0^\infty g(z) q_t(y,z)\,\mathrm{d}z + \int_0^t \int_0^\infty q_{t-s}(y,z) v(s,z)\, W(\mathrm{d}s,\mathrm{d}z) \\ &=: J(t,y) + I(t,y), \end{aligned} \tag{2.1}$$

for $t, y \in \mathbb{R}_+ := [0,\infty)$ where $q_t(y,z) = y^{-1} p_t^*(y,z) z$ is the transition density (from y to z) of a three-dimensional Bessel process, $p_t^*(y,z) = p_t(y-z) - p_t(y+z)$ is the transition density of Brownian motion killed at the origin and $p_t(y-z) = (2\pi t)^{-1/2} \exp(-(y-z)^2/2t)$ is the transition density of Brownian motion with the convention that $p_t(y) = 0$ for $t < 0$ for all y. The integral in the second term on the right hand side is a stochastic integral with respect to martingale measures in the sense of Walsh, see Appendix A for details. The function g is the initial condition which may be random but independent of the white noise.

The above integral equation is the mild form of the following stochastic partial differential equation (SPDE)

$$\begin{cases} \partial_t v(t,y) = \mathcal{L} v(t,y) + v(t,y) \dot{W}(t,y), & t, y \in \mathbb{R}_+, \\ v(0,y) = g(y), & y \in \mathbb{R}_+, \end{cases} \tag{2.2}$$

where $\mathcal{L} = \frac{1}{2}\Delta_y + y^{-1}\partial_y$ is the infinitesimal generator of a three-dimensional Bessel process and $\dot{W}$ denotes space-time white noise which formally is a generalised Gaussian random field with mean zero and covariance

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y).$$

The function g is the initial condition in the sense that

$$\lim_{t\downarrow 0}\int_0^\infty g(z)q_t(y,z)\,\mathrm{d}z = g(y),$$

since $\lim_{t\downarrow 0} q_t(y,z) = \delta_y(z)$, the Dirac delta function at y. We sometimes emphasise the initial data g and denote the solution by v^g. We also study the following integral equation for $t, x, y \in \mathbb{R}_+$

$$\begin{aligned} v(t,x,y) &= \frac{p_t^*(x,y)}{xy} + \int_0^t\int_0^\infty q_{t-s}(y,z)v(s,x,z)\,W(\mathrm{d}s,\mathrm{d}z) \\ &=: I(t,x,y) + J(t,x,y). \end{aligned} \tag{2.3}$$

Formally, $v(t,x,y)$ is the solution to (2.2) with $g = x^{-2}\delta_x$. From Lemma 2.2.2 below we see that $(xy)^{-1}p_t^*(x,y)$ is a continuous function of x and y over $\mathbb{R}_+\times\mathbb{R}_+$ and in particular it is equal to $\sqrt{2/\pi t^3}$ for x, y at the origin, hence equations (2.1) and (2.3) are well defined for x, $y = 0$.

Equation (2.1) is related to the mild form of the stochastic heat equation (SHE) on the half-line $\mathbb{R}_+$ with Dirichlet boundary conditions at 0:

$$u(t,y) = \int_0^\infty f(z)p_t^*(y,z)\,\mathrm{d}z + \int_0^t\int_0^\infty p_{t-s}^*(y,z)u(s,z)\,W(\mathrm{d}s,\mathrm{d}z), \tag{2.4}$$

for $t, y \in \mathbb{R}_+$. This mild form corresponds to (2.2) with $\mathcal{L} = \frac{1}{2}\Delta_y$ and with f in place of g. Observe that, for $y > 0$, setting $v(t,y) = u(t,y)/y$ and $g(y) = f(y)/y$ then dividing equation (2.4) through by y we obtain (2.1). When the initial condition $f = \delta_x$, $x \in \mathbb{R}_+$, we denote the corresponding solution by $u(t,x,y)$ to emphasise the position of the delta function. In this case, (2.4) now becomes

$$u(t,x,y) = p_t^*(x,y) + \int_0^t\int_0^\infty p_{t-s}^*(y,z)u(s,x,z)\,W(\mathrm{d}s,\mathrm{d}z). \tag{2.5}$$

The initial data is interpreted in the distributional sense, that is for all $\varphi \in C_c^\infty(\mathbb{R}_+)$,

$$\lim_{t\to 0}\int_{\mathbb{R}} \varphi(y)u(t,x,y)\,\mathrm{d}y = \varphi(x) \quad \text{almost surely.}$$

Setting $v(t,x,y) = u(t,x,y)/xy$ for x, $y > 0$ and dividing (2.5) through by xy we obtain equation (2.3).

Compare (2.3) with equation (1.19) in the introduction. On one hand, Dyson Brownian motion is a system of n Brownian motions conditioned in the sense of Doob to never collide. On the other hand, one can regard the three-dimensional Bessel process as Brownian motion conditioned to never hit the identically zero path. Moreover, contrast the transition density of Dyson Brownian motion $Q_t(\mathbf{x},\mathbf{y}) = \Delta(\mathbf{x})^{-1}p_n^*(t,\mathbf{x},\mathbf{y})\Delta(\mathbf{y})$ with that of the Bessel process $q_t(x,y) = x^{-1}p_t^*(x,y)y$; both are a product of two terms with one being a mixture of the heat kernel p_t and the other being a ratio of functions where the denominator is equal to 0 at some point. They thus present similar difficulties when attempting to prove certain continuity properties for them. With this in mind, (2.3) is a natural one-dimensional analogue of (1.19) and it serves as a guide to how one could tackle the latter which is our main motivation for studying it.

We are interested in the continuity of the solution to (2.1) and (2.3). It is well known that the solution to the stochastic heat equation is continuous in space (and time) almost surely and so it is easy to see that the function $y \mapsto v^g(t,y) = u^f(t,y)/y$ is continuous on $(0,\infty)$ and likewise for $y \mapsto v(t,x,y)$. However, it is not immediately obvious that v is continuous at the origin and the main contribution of this chapter is that this is indeed the case and there exists a unique solution to equations (2.1) and (2.3) which has a version that is jointly continuous in time and space, see Theorem 2.1.1. The key to the proof are certain continuity estimates for the kernel q_t, see Proposition 2.2.4.

An immediately corollary to the continuity result is that the solution to equation (2.4) has a derivative at the origin since the limit $\lim_{y\to 0} v(t,y) = \lim_{y\to 0} u(t,y)/y$ exists by the continuity of v and that $u(t,0) = 0$.

2.1.1 Main Result

Let us first set up the probability space. Let $\mathscr{B}_b(\mathbb{R})$ be the collection of Borel measurable subsets of $\mathbb{R}$ with finite Lebesgue measure and let $W = \big(W_t(A), t \geq 0, A \in \mathscr{B}_b(\mathbb{R})\big)$ be space-time white noise on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$ such that W is $\mathscr{F}_t$-adapted and $W_t(A) - W_s(A)$ is independent of $\mathscr{F}_s$ for all $A \in \mathscr{B}_b(\mathbb{R})$. From now on we fix this filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. We use $\mathbb{E}$ to denote the expectation with respect to $\mathbb{P}$ and for $p \geq 1$, $\|\cdot\|_p = (\mathbb{E}[|\cdot|^p])^{1/p}$ denotes the $L^p(\Omega)$ norm. Throughout $c_p \leq 2\sqrt{p}$, $p > 2$, $c_2 = 1$ is the constant appearing in the Burkholder–Davis–Gundy inequality. We denote the error function by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, \mathrm{d}t.$$

The main result of the chapter is the following

Theorem 2.1.1. *(a) Suppose that g is $\mathscr{F}_0$-measurable and satisfies for all $p \geq 2$*

$$\sup_{x\in\mathbb{R}_+} \|g(x)\|_p \leq K_{p,g} < \infty. \tag{2.6}$$

Then there exists a solution $\big(v(t,y), (t,y) \in \mathbb{R}_+ \times \mathbb{R}_+\big)$ to equation (2.1) that is unique (in the sense of versions) in the class of random fields $\big(f(t,y), (t,y) \in \mathbb{R}_+ \times \mathbb{R}_+\big)$ that satisfy $\sup_{(t,y)\in[0,T]\times\mathbb{R}_+} \|f(t,y)\|_p < \infty$ for all $T > 0$. The solution satisfies for all $p \geq 2$ and $(t,y) \in \mathbb{R}_+ \times \mathbb{R}_+$

$$\|v(t,y)\|_p^2 \leq 2K_{p,g}^2 e^{9c_p^4 t/4}\big(1 + \operatorname{erf}(3c_p^2 t^{1/2}/2)\big). \tag{2.7}$$

Moreover, v has a version such that $(t,y) \mapsto v(t,y)$ is locally Hölder continuous on $(0,\infty)\times\mathbb{R}_+$ with index $\alpha < 1/2$ in space and index $\alpha < 1/4$ in time.

(b) There exists a solution $\big(v(t,x,y),(t,x,y)\in(0,\infty)\times\mathbb{R}_+\times\mathbb{R}_+\big)$ to (2.3) that is unique in the class of random fields f such that $\int_0^t\int_0^\infty q_{t-s}(y,z)^2\|f(s,z)\|_2^2\,\mathrm{d}z\mathrm{d}s<\infty$ for all $(t,y)\in\mathbb{R}_+\times\mathbb{R}_+$. The solution satisfies for all $p\geq 2$ and $(t,x,y)\in(0,\infty)\times\mathbb{R}_+\times\mathbb{R}_+$,

$$\|v(t,x,y)\|_p^2 \leq 2\left(\frac{p_t^*(x,y)}{xy}\right)^2 \mathbb{E}_{x,y;t}^{X,X'}\big[\exp\big(2c_p^2L_t(X-X')\big)\big], \tag{2.8}$$

where X, X' are two independent copies of a three-dimensional Bessel bridge starting from x at time zero and ending in y at time t, $\mathbb{E}_{x,y;t}^{X,X'}$ denotes the expectation with respect to the joint law of the bridges and $L_t(X-X')$ is the local time at 0 of the difference of the two bridges.

Moreover, v has a version such that $(t,x,y)\mapsto v(t,x,y)$ is locally Hölder continuous on $(0,\infty)\times\mathbb{R}_+\times\mathbb{R}_+$ with index $\alpha<1/2$ in space and index $\alpha<1/4$ in time.

The outline for this chapter is as follows. In the next section we first derive some estimates for the kernel q_t which are central to the proof of the main result. Next, we introduce local times of Bessel bridges and derive a formula relating the moments of local times of differences of such bridges to integrals involving products of q_t at different space-time points which will be used in the proof of Theorem 2.1.1(b). The existence, uniqueness and moment estimates part of Theorem 2.1.1(a) and (b) is proved in Section 2.3. The final section of the chapter is devoted to proving the continuity of the solution.

2.2 Preliminaries

2.2.1 Estimates on q_t

We first prove the following bounds on integrals of the square of q_t.

Lemma 2.2.1. *For any $x\in\mathbb{R}_+$*

$$\int_0^\infty q_s^2(x,y)\,\mathrm{d}y \leq \frac{C_1}{\sqrt{s}},$$

$$\int_0^t\int_0^\infty q_s^2(x,y)\,\mathrm{d}y\mathrm{d}s \leq C_2\sqrt{t},$$

where $C_1 := \frac{3}{4\sqrt{\pi}}$ and $C_2 := \frac{3}{2\sqrt{\pi}}$.

Proof. Since the integrand is an even function we can replace the integral over $[0,\infty)$ with an integral over $\mathbb{R}$ times a factor of $1/2$. Recall that $q_t(x,y) = x^{-1}\big(p_t(x-y)-p_t(x+y)\big)y$, then

$$\begin{aligned}
\int_0^\infty q_s^2(x,y)\,\mathrm{d}y &= \frac{1}{4\pi s}\frac{1}{x^2}\int_{\mathbb{R}} y^2\big(e^{-(x-y)^2/s} - 2e^{-(x^2+y^2)/s} + e^{-(x+y)^2/s}\big)\,\mathrm{d}y \\
&= \frac{1}{4\sqrt{\pi s}}\frac{1}{x^2}\Big(x^2 + \frac{s}{2} - 2e^{-x^2/s}\frac{s}{2} + x^2 + \frac{s}{2}\Big) \\
&\le \frac{1}{4\sqrt{\pi s}}\frac{1}{x^2}\Big(2x^2 + s - s\Big(1 - \frac{x^2}{s}\Big)\Big) \\
&= \frac{3}{4\sqrt{\pi s}},
\end{aligned}$$

where we have used the fact that $e^{-x} \ge 1 - x$ for all $x \in \mathbb{R}$. The second inequality now follows from the first by a simple integration. □

The next lemma makes clear that $q_t(x,y)$ is a continuous function of x, y over $\mathbb{R}_+ \times \mathbb{R}_+$.

Lemma 2.2.2. *For all $t > 0$ and x, $y \in \mathbb{R}_+$, we have*

$$q_t(x,y) = \frac{y^2}{\sqrt{2\pi t^3}} e^{-(x^2+y^2)/2t} \int_{-1}^1 e^{\theta xy/t}\,\mathrm{d}\theta.$$

Proof. Since $\frac{2t}{xy}\sinh\left(\frac{xy}{t}\right) = \int_{-1}^1 e^{\theta xy/t}\,\mathrm{d}\theta$, we have

$$\begin{aligned}
q_t(x,y) &= \frac{1}{\sqrt{\pi}}\frac{1}{t}\Big(\frac{y}{x}\Big)^{1/2} y e^{-(x^2+y^2)/2t}\sqrt{\frac{2t}{xy}}\Big(\frac{e^{xy/t} - e^{-xy/t}}{2}\Big) \\
&= \frac{1}{\sqrt{\pi}}\frac{1}{t}\Big(\frac{y}{x}\Big)^{1/2} y e^{-(x^2+y^2)/2t}\sqrt{\frac{2t}{xy}}\sinh\Big(\frac{xy}{t}\Big) \\
&= \frac{y^2}{\sqrt{2\pi t^3}} e^{-(x^2+y^2)/2t}\int_{-1}^1 e^{\theta xy/t}\,\mathrm{d}\theta.
\end{aligned}$$

□

Lemma 2.2.3. *There exist finite constants C, $C' > 0$ such that for all $(t,x,y) \in (0,\infty) \times \mathbb{R}_+ \times \mathbb{R}_+$*

$$\frac{\partial q_t}{\partial x}(x,y) \le \frac{C}{\sqrt{t}} q_{2t}(x,y),$$

and

$$\frac{\partial q_t}{\partial t}(x,y) \le \frac{C'}{t} q_{2t}(x,y).$$

Proof. By Lemma 2.2.2,

$$q_t(x,y) = \frac{y^2}{\sqrt{2\pi t^3}}\int_{-1}^1 e^{-((\theta y - x)^2 + (1-\theta^2)y^2)/2t}\,\mathrm{d}\theta.$$

Differentiating and using the fact that $\sup_{x\in\mathbb{R}} xe^{-x^2} < \infty$ gives

$$\begin{aligned}\frac{\partial q_t}{\partial x}(x,y) &= \frac{2y^2}{\sqrt{2\pi t^4}}\int_{-1}^{1}\frac{(\theta y - x)}{2\sqrt{t}}e^{-(\theta y-x)^2/2t}e^{-(1-\theta^2)y^2/2t}\,\mathrm{d}\theta\\ &\le \frac{Cy^2}{\sqrt{2\pi t^4}}\int_{-1}^{1}e^{-(\theta y-x)^2/4t}e^{-(1-\theta^2)y^2/4t}\,\mathrm{d}\theta\\ &= \frac{C}{\sqrt{t}}q_{2t}(x,y),\end{aligned}$$

for some constant $C > 0$.

Denote $A_\theta = (\theta y - x)^2 + (1-\theta^2)y^2$ then differentiating with respect to t gives

$$\frac{\partial q_t}{\partial t}(x,y) = \frac{1}{t}\frac{2y^2}{\sqrt{2\pi t^3}}\int_{-1}^{1}e^{-A_\theta/2t}\Big(\frac{A_\theta}{4t} - \frac{3}{4}\Big)\,\mathrm{d}\theta.$$

Then in the same manner as above, there exists a constant $C' > 0$ such that

$$\frac{\partial q_t}{\partial t}(x,y) \le \frac{C'}{t}\frac{y^2}{\sqrt{4\pi t^3}}\int_{-1}^{1}e^{-A_\theta/4t}\,\mathrm{d}\theta = \frac{C'}{t}q_{2t}(x,y). \tag{2.9}$$

Note that in the above calculations we can differentiate under the integral sign because the integrand is bounded uniformly by the constant 1 and the region of integration is bounded and so we can appeal to Proposition 3.2.7. □

Proposition 2.2.4. *There exists a constant $C_3 = \frac{3}{\sqrt{2\pi}}$ such that for all x, $z \in \mathbb{R}_+$ and $t > 0$, we have*

$$\int_0^t\int_0^\infty \big(q_s(x,y) - q_s(z,y)\big)^2\,\mathrm{d}y\mathrm{d}s \le C_3|x-z|. \tag{2.10}$$

There exist a constant $C_4 > 0$ such that for all $0 < u \le t < \infty$ and $x \in \mathbb{R}_+$,

$$\int_0^u\int_0^\infty \big(q_{t-s}(x,y) - q_{u-s}(x,y)\big)^2\,\mathrm{d}y\mathrm{d}s \le C_4|t-u|^{1/2}, \tag{2.11}$$

and

$$\int_u^t\int_0^\infty q_{t-s}(x,y)^2\,\mathrm{d}y\mathrm{d}s \le C_2|t-u|^{1/2}, \tag{2.12}$$

where C_2 is the constant defined in Lemma 2.2.1.

First observe that q_t has the following scaling property:

$$q_t(x,y) = t^{-1/2}q_1(xt^{-1/2}, yt^{-1/2}).$$

The left hand side of the inequality (2.10) is bounded above by

$$\int_0^\infty\int_0^\infty \big(q_s(x,y) - q_s(z,y)\big)^2\,\mathrm{d}y\mathrm{d}s = \int_0^\infty\frac{1}{\sqrt{s}}\int_0^\infty \big(q_1(xs^{-1/2},y) - q_1(zs^{-1/2},y)\big)^2\,\mathrm{d}y\mathrm{d}s, \tag{2.13}$$

where we have changed the integration region to $[0,\infty)$ in the time integral which results in an upper bound due to the positivity of the integrand. The equality follows from the scaling property and the change of variables $ys^{-1/2} \mapsto y$. Inequality (2.10) now follows from (2.13) and Lemma 2.2.5 below.

Lemma 2.2.5. *Suppose a function $R(\mathbf{x}, y) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfies for some constants c_1, $c_2 > 0$*

$$\int_{\mathbb{R}} \big(R(\mathbf{x}, y) - R(\mathbf{z}, y)\big)^2 \, \mathrm{d}y \le \min(c_1, c_2|\mathbf{x}-\mathbf{z}|^2), \tag{2.14}$$

for any $\mathbf{x}$, $\mathbf{z} \in \mathbb{R}^d$ where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$, then

$$\int_0^\infty \frac{1}{\sqrt{t}} \int_{\mathbb{R}} \big(R(\mathbf{x}/\sqrt{t}, y) - R(\mathbf{z}/\sqrt{t}, y)\big)^2 \, \mathrm{d}y\mathrm{d}t \le C|\mathbf{x}-\mathbf{z}|,$$

with $C = 2\sqrt{c_1 c_2}$.

Proof.

$$\begin{aligned}\int_0^\infty \frac{1}{\sqrt{t}} \int_{\mathbb{R}} & \big(R(\mathbf{x}/\sqrt{t}, y) - R(\mathbf{z}/\sqrt{t}, y)\big)^2 \, \mathrm{d}y\mathrm{d}t \\ & \le \int_0^{\frac{c_2}{c_1}|\mathbf{x}-\mathbf{z}|^2} \frac{c_1}{\sqrt{t}} \, \mathrm{d}t + \int_{\frac{c_2}{c_1}|\mathbf{x}-\mathbf{z}|^2}^\infty \frac{c_2}{t^{3/2}} |\mathbf{x}-\mathbf{z}|^2 \, \mathrm{d}t = C|\mathbf{x}-\mathbf{z}|.\end{aligned}$$

□

Proof of Proposition 2.2.4. We first prove inequality (2.10). By (2.13) we only need to verify the hypothesis of Lemma 2.2.5 for $d = 1$ and $R(x, y) = q_1(x, y)$. On the one hand, by Lemma 2.2.1 we have

$$\int_0^\infty \big(q_1(x, y) - q_1(z, y)\big)^2 \, \mathrm{d}y \le 2 \int_0^\infty q_1(x, y)^2 + q_1(z, y)^2 \, \mathrm{d}y \le 4C_1.$$

On the other hand, let $q_1'(\eta, y) = \frac{\partial}{\partial x} q_1(x, y)|_{x=\eta}$ be the derivative of q_1 in the first variable, then by Minkowski's integral inequality [Kal02, Corollary 1.30], assuming without loss of generality that $x < z$, we have

$$\begin{aligned}\left(\int_0^\infty \big(q_1(x, y) - q_1(z, y)\big)^2 \, \mathrm{d}y\right)^{1/2} &= \left(\int_0^\infty \left(\int_x^z q_1'(\eta, y) \, \mathrm{d}\eta\right)^2 \mathrm{d}y\right)^{1/2} \\ &\le \int_x^z \left(\int_0^\infty q_1'(\eta, y)^2 \, \mathrm{d}y\right)^{1/2} \mathrm{d}\eta \\ &\le \sup_{\eta \in \mathbb{R}_+} \|q_1'(\eta, \cdot)\|_{L^2(\mathbb{R}_+, \mathrm{d}y)} |x - z|.\end{aligned}$$

Thus, it remains to show that $\sup_{\eta\in\mathbb{R}_+}\|q_1'(\eta,\cdot)\|_{L^2(\mathbb{R}_+,\mathrm{d}y)}<\infty$. Differentiating gives

$$\begin{aligned}q_1'(\eta,y)&=\frac{y}{\eta}\Big((y-\eta)p_1(\eta-y)+(y+\eta)p_1(\eta+y)\Big)-\frac{y}{\eta^2}\Big(p_1(\eta-y)-p_1(\eta+y)\Big)\\&=\frac{1}{\eta^2}\Big((-y+\eta y^2-\eta^2y)p_1(\eta-y)+(y+\eta y^2+\eta^2y)p_1(\eta+y)\Big).\end{aligned}$$

We split $\|q_1'(\eta,\cdot)\|_{L^2(\mathbb{R}_+)}=\frac{1}{2}\|q_1'(\eta,\cdot)\|_{L^2(\mathbb{R})}$ into three parts $\mathrm{I}+\mathrm{II}+2\mathrm{III}$ where

$$\begin{aligned}\mathrm{I}&=\frac{1}{4\eta^4\sqrt{\pi}}\int_{\mathbb{R}}(-y+\eta y^2-\eta^2y)^2p_{1/2}(\eta-y)\,\mathrm{d}y\\&=\frac{1}{16\eta^4\sqrt{\pi}}(2\eta^4-\eta^2+2).\end{aligned}$$

Note that in the first equality above we have used the fact that $p_1(\eta-y)^2=\frac{1}{2\sqrt{\pi}}p_{1/2}(\eta-y)$. By symmetry,

$$\mathrm{II}=\frac{1}{4\eta^4\sqrt{\pi}}\int_{\mathbb{R}}(y+\eta y^2+\eta^2y)^2p_{1/2}(\eta+y)\,\mathrm{d}y=\frac{1}{16\eta^4\sqrt{\pi}}(2\eta^4-\eta^2+2)$$

also and

$$\begin{aligned}\mathrm{III}&=\frac{1}{4\eta^4}\int_{\mathbb{R}}(y+\eta y^2+\eta^2y)(-y+\eta y^2-\eta^2y)p_{1/2}(y)p_{1/2}(\eta)\,\mathrm{d}y\\&=-\frac{1}{16\eta^4\sqrt{\pi}}(2\eta^4+\eta^2+2)e^{-\eta^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|q_1'(\eta,\cdot)\|_{L^2(\mathbb{R}_+,\mathrm{d}y)}&=\frac{1}{8\eta^4\sqrt{\pi}}\big((2\eta^4-\eta^2+2)-(2\eta^4+\eta^2+2)e^{-\eta^2}\big)\\&=\frac{1}{8\sqrt{\pi}}\Big(\frac{2(1-e^{-\eta^2})}{\eta^4}+2(1-e^{-\eta^2})-\frac{1}{\eta^2}-\frac{e^{-\eta^2}}{\eta^2}\Big)\\&\le\frac{1}{8\sqrt{\pi}}\Big(2(1-e^{-\eta^2})+\frac{1-e^{-\eta^2}}{\eta^2}\Big)\\&\le\frac{1}{8\sqrt{\pi}}\big(2(1-e^{-\eta^2})+1\big)\\&\le\frac{3}{8\sqrt{\pi}},\end{aligned}$$

where in the first and second inequality we have used the fact that $1-e^{-x}\le x$ for all $x\in\mathbb{R}$. Thus, we have shown that (2.14) holds with $R(x,y)=q_1(x,y)$ and the desired result follows from Lemma 2.2.5. One can check that the constant appearing on the right hand side of (2.10) is equal to $2\big(4C_13(8\sqrt{\pi})^{-1}\big)^{1/2}=\frac{3}{\sqrt{2\pi}}=:C_3$.

We now turn our attention to proving inequality (2.11). Suppose that $t=u+h$ for some $h>0$. Making the change of variables $u-s=hs'$ and using the scaling property of

q_t the left hand side of (2.11) is equal to

$$\begin{aligned}
h\int_0^{u/h}&\int_0^\infty \big(q_{h(s'+1)}(x,y)-q_{hs'}(x,y)\big)^2\,\mathrm{d}y\mathrm{d}s' \\
&= \int_0^{u/h}\int_0^\infty \big(q_{s'+1}(xh^{-1/2},yh^{-1/2})-q_{s'}(xh^{-1/2},yh^{-1/2})\big)^2\,\mathrm{d}y\mathrm{d}s' \\
&\le h^{1/2}\int_0^\infty\int_0^\infty \big(q_{s'+1}(xh^{-1/2},y')-q_{s'}(xh^{-1/2},y')\big)^2\,\mathrm{d}y'\mathrm{d}s'.
\end{aligned}$$

Therefore, it suffices to show that

$$\int_0^\infty\int_0^\infty \big(q_{s+1}(x,y)-q_s(x,y)\big)^2\,\mathrm{d}y\mathrm{d}s<\infty$$

uniformly for $x\in\mathbb{R}_+$. By Lemma 2.2.1

$$\begin{aligned}
\int_0^1\int_0^\infty &\big(q_{s+1}(x,y)-q_s(x,y)\big)^2\,\mathrm{d}y\mathrm{d}s \\
&\le 2\int_0^1\int_0^\infty q_{s+1}(x,y)^2\,\mathrm{d}y\mathrm{d}s+2\int_0^1\int_0^\infty q_s(x,y)^2\,\mathrm{d}y\mathrm{d}s \\
&\le 2\sqrt{2}C_2.
\end{aligned}$$

Hence, it remains to estimate the contribution of the integral over the region $[1,\infty)$. From Lemma 2.2.3, we know that $\frac{\partial q_t}{\partial t}(x,y)\le Ct^{-1}q_{2t}(x,y)$ for some constant $C>0$ and so by Minkowski's integral inequality and Lemma 2.2.1 we have

$$\begin{aligned}
\left(\int_0^\infty \big(q_{s+1}(x,y)-q_s(x,y)\big)^2\,\mathrm{d}y\right)^{1/2} &= \left(\int_0^\infty\left(\int_s^{s+1}\frac{\partial}{\partial r}q_r(x,y)\,\mathrm{d}r\right)^2\mathrm{d}y\right)^{1/2} \\
&\le \int_s^{s+1}\left(\int_0^\infty \frac{C^2}{r^2}q_{2r}(x,y)^2\,\mathrm{d}y\right)^{1/2}\mathrm{d}r \\
&\le C'\int_s^{s+1}\frac{1}{r^{5/4}}\,\mathrm{d}r \\
&\le \frac{C'}{s^{5/4}},
\end{aligned}$$

where the constant $C'>0$ is independent of s and x. Thus,

$$\int_1^\infty\int_0^\infty \big(q_{s+1}(x,y)-q_s(x,y)\big)^2\,\mathrm{d}y\mathrm{d}s\le C'^2\int_1^\infty s^{-5/2}\,\mathrm{d}s<\infty,$$

which completes the prove of inequality (2.11).

Finally, making the change of variable $s'=t-s$, we have

$$\int_u^t\int_0^\infty q_{t-s}(x,y)^2\,\mathrm{d}y\mathrm{d}s=\int_0^{t-u}\int_0^\infty q_{s'}(x,y)^2\,\mathrm{d}y\mathrm{d}s',$$

then applying Lemma 2.2.1 gives inequality (2.12) which completes the proof. □

2.2.2 Local Time of Difference of Bessel Bridges

Let $f \in C_c^\infty(\mathbb{R})$, a smooth function with compact support such that $f \geq 0$, f is even and $\int_\mathbb{R} f(y)\ \mathrm{d}y = 1$. Define for $\varepsilon > 0$, $f_\varepsilon(y) := \varepsilon^{-1} f(y\varepsilon^{-1})$ and for two independent three-dimensional Bessel bridges X, X' starting from x and x' at time 0 and ending at y and y' at time t repectively, define

$$L_t^{(\varepsilon)} := L_t^{(\varepsilon)}(X - X') := \int_0^t f_\varepsilon(X_s - X'_s)\ \mathrm{d}s.$$

Let $\mathbb{E}^X_{x,y;t}$ denote the expectation with respect to the law of the Bessel bridge X then $L_t^{(\varepsilon)}$ is an approximation to the local time $L_t = L_t(X - X')$ at 0 of the difference of the bridges in the following sense.

Lemma 2.2.6. *For all $p \geq 1$ there exists a constant $C := C(p) > 0$ such that for all $t > 0$*

$$\sup_{x,x',y,y' \in \mathbb{R}_+} p_t^*(x,y) p_t^*(x',y') \mathbb{E}^X_{x,y;t} \mathbb{E}^{X'}_{x',y';t} \left[|L_t^{(\varepsilon)} - L_t|^p \right] \leq C t^{p/4-1} \varepsilon^{p/2}.$$

Proof. By the occupation times formula [RY99, Chapter VI, Corollary 1.6]

$$L_t^{(\varepsilon)} = \int_\mathbb{R} f_\varepsilon(a) L_t^a\ \mathrm{d}a,$$

where L_t^a is the local time at a of the difference of the bridges. When $a = 0$ we simply write $L_t^0 = L_t$. For brevity we write $\mathbb{E}^X_{x,y;t} \mathbb{E}^{X'}_{x',y';t}[|\cdot|^p] = \|\cdot\|^p_{L^p(X,X')}$. By the assumptions on f, $\int_\mathbb{R} f_\varepsilon(a)\ \mathrm{d}a = 1$ and so

$$\begin{aligned} \|L_t^{(\varepsilon)} - L_t\|_{L^p(X,X')} &= \left\| \int_\mathbb{R} f_\varepsilon(a)(L_t^a - L_t)\ \mathrm{d}a \right\|_{L^p(X,X')} \\ &= \left\| \varepsilon \int_\mathbb{R} f_\varepsilon(\varepsilon a)(L_t^{\varepsilon a} - L_t)\ \mathrm{d}a \right\|_{L^p(X,X')} \\ &\leq \int_\mathbb{R} f(a) \|L_t^{\varepsilon a} - L_t\|_{L^p(X,X')}\ \mathrm{d}a \end{aligned} \tag{2.15}$$

by Minkowski's integral inequality. We shall bound the last line of the above by using the $L^p(X,X')$ Hölder continuity of the local times which we will prove now.

Decompose the local time into two parts: $L_t^a = L^a_{[0,t/2]} + L^a_{[t/2,t]}$, where $L^a_{[0,t/2]}$ denotes the local time over the time period from 0 to $t/2$ and likewise for the other term. Let's consider $L^a_{[0,t/2]}$ first. Note that

$$\begin{aligned} \frac{\mathrm{d}\mathbb{P}^X_{x,y;t}}{\mathrm{d}\mathbb{P}_x} &= \frac{\mathrm{d}\mathbb{P}^X_{x,y;t}}{\mathrm{d}\mathbb{P}^{\mathrm{Bes}}_x} \times \frac{\mathrm{d}\mathbb{P}^{\mathrm{Bes}}_x}{\mathrm{d}\mathbb{P}_x} = \frac{q_{t/2}(X_{t/2}, y)}{q_t(x,y)} \frac{X_{t/2}}{x} \\ &= \frac{p^*_{t/2}(X_{t/2}, y)}{p^*_t(x,y)} \quad \text{on } \mathscr{F}^X_{t/2}, \end{aligned} \tag{2.16}$$

where $\mathbb{P}_x^{\mathrm{Bes}}$ and $\mathbb{P}_x$ denote the law of a three-dimensional Bessel process started at x and the law of Brownian motion started at x respectively. Let $D_t(x,y) := (\pi t)^{-\frac{1}{2}} p_t^*(x,y)^{-1}$ then it is easy to see that $p_{t/2}^*(X_{t/2},y)/p_t^*(x,y) \leq D_t(x,y)$. Then, for any $a, b \in [0,\infty)$

$$\begin{aligned}\|L^a_{[0,t/2]} - L^b_{[0,t/2]}\|^p_{L^p(X,X')} &\leq D_t(x,y)D_t(x',y')\mathbb{E}_x\mathbb{E}_{x'}[|L^a_{[0,t/2]} - L^b_{[0,t/2]}|^p] \\ &= D_t(x,y)D_t(x',y')\mathbb{E}_{x-x'}[|L^a_{[0,t/2]} - L^b_{[0,t/2]}|^p],\end{aligned}$$

since a difference of two Brownian motions started from x and x' is in law a Brownian motion started from $x - x'$.

For $L^a_{[t/2,t]}$, we can by time reversal, consider it as the local time at a of the difference of two independent Bessel bridges starting from y, y' and ending at x, x' respectively. Hence, by the same reasoning as for $L^a_{[0,t/2]}$ we have

$$\|L^a_{[t/2,t]} - L^b_{[t/2,t]}\|^p_{L^p(X,X')} \leq D_t(y,x)D_t(y',x')\mathbb{E}_{y-y'}[|L^a_{[0,t/2]} - L^b_{[0,t/2]}|^p].$$

By the L^p Hölder continuity of Brownian local times for every p (see [RY99, Chapter VI, Corollary 1.8], there exists a constant $C_p > 0$ such that for all $z \in \mathbb{R}$ and $t > 0$

$$\mathbb{E}_z[|L^a_t - L^b_t|^p] \leq C_p t^{p/4}|a-b|^{p/2}.$$

Using this and combining the contributions from both $L^a_{[0,t/2]}$ and $L^a_{[t/2,t]}$ and noting that $D_t(x,y) = D_t(y,x)$, we have

$$\|L^a_t - L^b_t\|^p_{L^p(X,X')} \leq 2^{1-p/4}C_p t^{p/4} D_t(x,y)D_t(x',y')|a-b|^{p/2}. \tag{2.17}$$

Finally, by (2.15)

$$\begin{aligned}&p_t^*(x,y)p_t^*(x',y')\|L_t^{(\varepsilon)} - L_t\|^p_{L^p(X,X')} \\ &\qquad \leq 2^{1-p/4}C_p t^{p/4} p_t^*(x,y)p_t^*(x',y')D_t(x,y)D_t(x',y')\varepsilon^{p/2}\left(\int_{\mathbb{R}} f(a)|a|^{1/2}\,\mathrm{d}a\right)^p \\ &\qquad = 2^{1-p/4}CC_p t^{p/4-1}\varepsilon^{p/2},\end{aligned}$$

where $C^{1/p} := \int_{\mathbb{R}} f(a)|a|^{1/2}\,\mathrm{d}a < \infty$ as $f \in C_c^\infty(\mathbb{R})$. $\square$

The next result relates integrals of products of q_t with moments of the local time L_t. It will be used together with the bound on the exponential moments of L_t (Lemma 2.2.9) to estimate the pth moments of the solution to (2.3).

Proposition 2.2.7. *The following holds for all x, $y \in \mathbb{R}_+$, $t > 0$ and $k \geq 1$.*

$$\mathbb{E}^X_{x,y;t}\mathbb{E}^{X'}_{x,y;t}[(L_t)^k] = k!\int_{\Delta_k(t)}\int_{\mathbb{R}^k_+} R_k(\mathbf{s},\mathbf{z};t,x,y)^2\prod_{i=1}^k \mathrm{d}z_i\mathrm{d}s_i, \tag{2.18}$$

where for $\mathbf{s} := (s_1, \ldots, s_k) \in \Delta_k(t) := \{0 < s_k < \ldots < s_1 < t\}$ *and* $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{R}_+^k$,

$$\begin{aligned}
R_k(\mathbf{s}, \mathbf{z}; t, x, y) &:= \frac{q_{t-s_1}(y, z_1)}{q_t(y, x)} \prod_{i=2}^{k} q_{s_{i-1}-s_i}(z_{i-1}, z_i) q_{s_k}(z_k, x) \\
&= \frac{p^*_{t-s_1}(y, z_1)}{p^*_t(y, x)} \prod_{i=2}^{k} p^*_{s_{i-1}-s_i}(z_{i-1}, z_i) p^*_{s_k}(z_k, x).
\end{aligned}$$

Proof. Fix x, x', y, y' and t. Let X, X' be independent three-dimensional Bessel bridges from y to x and y' to x' repectively. We first prove by induction on k that

$$\begin{aligned}
&\mathbb{E}^X_{y,x;t} \mathbb{E}^{X'}_{y',x';t} \left[\int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} \prod_{i=1}^{k} f_\varepsilon(X_{t-s_i} - X'_{t-s_i}) \, \mathrm{d}s_i \right] \\
&\qquad = \int_{\Delta_k(t)} \int_{\mathbb{R}_+^k \times \mathbb{R}_+^k} R_k(\mathbf{s}, \mathbf{z}; t, x, y) R_k(\mathbf{s}, \mathbf{z}'; t, x', y') \prod_{i=1}^{k} f_\varepsilon(z_i - z'_i) \, \mathrm{d}z_i \mathrm{d}z'_i \mathrm{d}s_i.
\end{aligned} \tag{2.19}$$

The result follows upon sending $\varepsilon \to 0$ and noting that X_{t-s} is in law a Bessel bridge from x to y.

Firstly, we have by Fubini's theorem and using the transition density of X_{t-s} and X'_{t-s} that

$$\begin{aligned}
&\mathbb{E}^X_{y,x;t} \mathbb{E}^{X'}_{y',x';t} \left[\int_0^t f_\varepsilon(X_{t-s} - X'_{t-s}) \, \mathrm{d}s \right] \\
&\quad = \int_0^t \int_{\mathbb{R}_+ \times \mathbb{R}_+} \frac{q_{t-s}(y, z) q_s(z, x)}{q_t(y, x)} f_\varepsilon(z - z') \frac{q_{t-s}(y', z') q_s(z', x')}{q_t(y', x')} \, \mathrm{d}z \mathrm{d}z' \mathrm{d}s \\
&\quad = \int_0^t \int_{\mathbb{R}_+ \times \mathbb{R}_+} R_1(s, z; t, x, y) f_\varepsilon(z - z') R_1(s, z'; t, x', y') \, \mathrm{d}z \mathrm{d}z' \mathrm{d}s.
\end{aligned}$$

Hence, (2.19) is true for $k = 1$. Assume that it is true for $k-1$, $k \geq 2$. For the kth case, since $f_\varepsilon(X_{t-s_1} - X'_{t-s_1})$ is $\mathscr{F}_{t-s_1} \times \mathscr{F}'_{t-s_1}$ measurable, where $(\mathscr{F}_t)_{t \geq 0}$, $(\mathscr{F}')_{t \geq 0}$ are the filtrations generated by X, X' respectively, we have by the properties of conditional expectation that

$$\begin{aligned}
&\mathbb{E}^X_{y,x;t} \mathbb{E}^{X'}_{y',x';t} \left[\int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} \prod_{i=1}^{k} f_\varepsilon(X_{t-s_i} - X'_{t-s_i}) \, \mathrm{d}s_i \right] \\
&\quad = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} \mathbb{E}^X_{y,x;t} \mathbb{E}^{X'}_{y',x';t} \left[\prod_{i=1}^{k} f_\varepsilon(X_{t-s_i} - X'_{t-s_i}) \right] \prod_{i=1}^{k} \mathrm{d}s_i \\
&\quad = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} \mathbb{E}^X_{y,x;t} \mathbb{E}^{X'}_{y',x';t} \Bigg[f_\varepsilon(X_{t-s_1} - X'_{t-s_1}) \\
&\qquad \times \mathbb{E}^X_{y,x;t} \mathbb{E}^{X'}_{y',x';t} \left[\prod_{i=2}^{k} f_\varepsilon(X_{t-s_i} - X'_{t-s_i}) \Big| \mathscr{F}_{t-s_1} \times \mathscr{F}'_{t-s_1} \right] \Bigg] \prod_{i=1}^{k} \mathrm{d}s_i
\end{aligned} \tag{2.20}$$

By the Markov property, the conditional expectation in the last line above is equal to

$$\mathbb{E}^X_{z_1,x;s_1}\mathbb{E}^{X'}_{z_1',x';s_1}\Big[\prod_{i=2}^k f_\varepsilon(X_{s_1-s_i}-X'_{s_1-s_i})\Big]\Big|_{z_1=X_{t-s_1},z_1'=X'_{t-s_1}}.$$

Using the transition density of X_{t-s_1} and X'_{t-s_1} and the induction hypothesis, the last line of (2.20) becomes

$$\begin{aligned}
&\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1}}\int_{\mathbb{R}_+\times\mathbb{R}_+} R_1(s_1,z_1;t,x,y)R_1(s_1,z_1';t,x',y')f_\varepsilon(z_1-z_1')\\
&\qquad\times\mathbb{E}^X_{z_1,x;s_1}\mathbb{E}^{X'}_{z_1',x';s_1}\Big[\prod_{i=2}^k f_\varepsilon(X_{s_1-s_i}-X'_{s_1-s_i})\Big]\,\mathrm{d}z_1\mathrm{d}z_1'\prod_{i=1}^k\mathrm{d}s_i\\
&=\int_0^t\int_{\mathbb{R}_+\times\mathbb{R}_+} R_1(s_1,z_1;t,x,y)R_1(s_1,z_1';t,x',y')f_\varepsilon(z_1-z_1')\\
&\qquad\times\mathbb{E}^X_{z_1,x;s_1}\mathbb{E}^{X'}_{z_1',x';s_1}\Big[\int_0^{s_1}\cdots\int_0^{s_{k-1}}\prod_{i=2}^k f_\varepsilon(X_{s_1-s_i}-X'_{s_1-s_i})\,\mathrm{d}s_i\Big]\,\mathrm{d}z_1\mathrm{d}z_1'\mathrm{d}s_1\\
&=\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1}}\int_{\mathbb{R}_+^k\times\mathbb{R}_+^k} R_1(s_1,z_1;t,x,y)R_1(s_1,z_1';t,x',y')\prod_{i=1}^k f_\varepsilon(z_i-z_i')\\
&\qquad\times R_{k-1}(\mathbf{s}_{2:k},\mathbf{z}_{2:k};s_1,z_1,y)R_{k-1}(\mathbf{s}_{2:k},\mathbf{z}'_{2:k};s_1,z_1',y')\,\prod_{i=1}^k\mathrm{d}z_i\mathrm{d}z_i'\mathrm{d}s_i,
\end{aligned}$$

where $\mathbf{s}_{2:k}=(s_2,\dots,s_k)\in\Delta_{k-1}(s_1)$ and $\mathbf{z}_{2:k}=(z_2,\dots,z_k)\in\mathbb{R}_+^{k-1}$. Equation (2.19) then follows since

$$R_k(\mathbf{s},\mathbf{z};t,x,y)=R_1(s_1,z_1;t,x,y)R_{k-1}(\mathbf{s}_{2:k},\mathbf{z}_{2:k};s_1,z_1,y).$$

This completes the induction.

Denote $P_k^{t,x,y}(\mathbf{s},\mathbf{z}):=p^*_{t-s_1}(y,z_1)\prod_{i=2}^k p^*_{s_{i-1}-s_i}(z_{i-1},z_i)p^*_{s_k}(z_k,x)$ then

$$R_k(\mathbf{s},\mathbf{z};t,x,y)=\frac{P_k^{t,x,y}(\mathbf{s},\mathbf{z})}{p^*_t(x,y)}.$$

Now set $x=x'$, $y=y'$ and multiply both sides of (2.19) by $p_t^*(x,y)^2$, then the right hand side of (2.19) becomes

$$\int_{\Delta_k(t)}\int_{\mathbb{R}_+^k} P_k^{t,x,y}(\mathbf{s},\mathbf{z})\big(F_\varepsilon * P_k^{t,x,y}(\mathbf{s},\cdot)\big)(\mathbf{z})\,\prod_{i=1}^k\mathrm{d}z_i\mathrm{d}s_i,$$

where $F_\varepsilon:=\prod_{i=1}^k f_\varepsilon$ and

$$\big(F_\varepsilon * P_k^{t,x,y}(\mathbf{s},\cdot)\big)(\mathbf{z})=\int_{\mathbb{R}_+^k} P_k^{t,x,y}(\mathbf{s},\mathbf{z}')\prod_{i=1}^k f_\varepsilon(z_i-z_i')\,\mathrm{d}z_i'.$$

Since $P_k^{t,x,y}(\mathbf{s},\mathbf{z})$ is bounded by a product of heat kernels, we have for all $\varepsilon > 0$ and making the change of variable $w_i = (z_i - z_i')/\varepsilon$

$$
\begin{aligned}
&\big(F_\varepsilon * P_k^{t,x,y}(\mathbf{s},\cdot)\big)(\mathbf{z}) \\
&\quad \le \int_{\mathbb{R}_+^k} p_{t-s_1}(y - z_1') \prod_{i=2}^k p_{s_{i-1}-s_i}(z_{i-1}' - z_i') p_{s_k}(z_k' - x) \prod_{i=1}^k f_\varepsilon(z_i - z_i')\, \mathrm{d}z_i' \\
&\quad \le \int_{\mathbb{R}^k} p_{t-s_1}\big(y - (z_1 - \varepsilon w_1)\big) \prod_{i=2}^k p_{s_{i-1}-s_i}\big((z_{i-1} - \varepsilon w_{i-1}) - (z_i - \varepsilon w_i)\big) \\
&\qquad\quad \times p_{s_k}(z_k - \varepsilon w_k - x) \prod_{i=1}^k f(w_i)\, \mathrm{d}w_i \\
&\quad \le (2\pi)^{-k/2} \|f\|_{L^1(\mathbb{R})}^k \frac{1}{\sqrt{t-s_1}} \prod_{i=2}^k \frac{1}{\sqrt{s_{i-1}-s_i}} \frac{1}{\sqrt{s_k}},
\end{aligned}
$$

where in the last inequality we used the elementary estimate $p_t(x-y) \le (2\pi t)^{-1/2}$ for all $x, y \in \mathbb{R}$. Using the Chapman–Kolmogorov equation for p_t we have for each $t > 0$ that

$$
\begin{aligned}
&\|f\|_{L^1(\mathbb{R})}^k \int_{\Delta_k(t)} \frac{1}{\sqrt{t-s_1}} \prod_{i=2}^k \frac{1}{\sqrt{s_{i-1}-s_i}} \frac{1}{\sqrt{s_k}} \int_{\mathbb{R}^k} P_k^{t,x,y}(\mathbf{s},\mathbf{z}) \prod_{i=1}^k \mathrm{d}z_i \mathrm{d}s_i \\
&\qquad\qquad \le \|f\|_{L^1(\mathbb{R})} p_t(x-y) \int_{\Delta_k(t)} \frac{1}{\sqrt{t-s_1}} \prod_{i=2}^k \frac{1}{\sqrt{s_{i-1}-s_i}} \frac{1}{\sqrt{s_k}} \prod_{i=1}^k \mathrm{d}s_i \\
&\qquad\qquad \le \|f\|_{L^1(\mathbb{R})} p_t(x-y) \frac{\pi^{(k+1)/2} t^{(k-1)/2}}{\Gamma\Big(\frac{k+1}{2}\Big)},
\end{aligned}
$$

where we have used Lemma 2.2.8 below to evaluate the time integral. Therefore, by the dominated convergence theorem

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\Delta_k(t)} \int_{\mathbb{R}_+^k} P_k^{t,x,y}(\mathbf{s},\mathbf{z}) \big(F_\varepsilon * P_k^{t,x,y}(\mathbf{s},\cdot)\big)(\mathbf{z}) \prod_{i=1}^k \mathrm{d}z_i \mathrm{d}s_i \\
&\qquad = \int_{\Delta_k(t)} \int_{\mathbb{R}_+^k} \lim_{\varepsilon\to 0} P_k^{t,x,y}(\mathbf{s},\mathbf{z}) \big(F_\varepsilon * P_k^{t,x,y}(\mathbf{s},\cdot)\big)(\mathbf{z}) \prod_{i=1}^k \mathrm{d}z_i \mathrm{d}s_i \\
&\qquad = \int_{\Delta_k(t)} \int_{\mathbb{R}_+^k} P_k^{t,x,y}(\mathbf{s},\mathbf{z})^2 \prod_{i=1}^k \mathrm{d}z_i \mathrm{d}s_i.
\end{aligned}
$$

In the last line above we used the fact that F_ε is an approximate delta function on $\mathbb{R}^k$ and since $\mathbf{z} \mapsto P_k^{t,x,y}(\mathbf{s},\mathbf{z})$ is continuous and vanishes at infinity, $(F_\varepsilon * P_k^{t,x,y}(\mathbf{s},\cdot))(\mathbf{z})$ converges to $P_k^{t,x,y}(\mathbf{s},\mathbf{z})$ uniformly in $\mathbf{z}$, see [Zua01, Theorem 2.1].

Finally, since the function $\prod_{i=1}^k f_\varepsilon(X_{t-s_i} - X'_{t-s_i})$ is symmetric with respect to permutations of the variables $s_1, \ldots, s_k$, its integral over $\Delta_k(t)$ can be replaced with an integral over $[0,t]^k$ times a factor of $(k!)^{-1}$. Hence, the left hand side of (2.19) (with $x = x'$,

$y = y'$) is equal to

$$\frac{1}{k!}\mathbb{E}^X_{y,x;t}\mathbb{E}^{X'}_{y,x;t}\left[\int_{[0,t]^k}\prod_{i=1}^k f_\varepsilon(X_{t-s_i} - X'_{t-s_i})\,\mathrm{d}s_i\right]$$
$$= \frac{1}{k!}\mathbb{E}^X_{x,y;t}\mathbb{E}^{X'}_{x,y;t}\left[\int_{[0,t]^k}\prod_{i=1}^k f_\varepsilon(X_{s_i} - X'_{s_i})\,\mathrm{d}s_i\right]$$
$$= \frac{1}{k!}\mathbb{E}^X_{x,y;t}\mathbb{E}^{X'}_{x,y;t}\big[(L_t^{(\varepsilon)})^k\big],$$

where in the first equality we used the fact that $\tilde{X}_s := X_{t-s}$ is in law a Bessel bridge starting from x and ending at y at time t. By Lemma 2.2.6,

$$\lim_{\varepsilon\to 0} p_t^*(x,y)^2\mathbb{E}^X_{x,y;t}\mathbb{E}^{X'}_{x,y;t}\big[(L_t^{(\varepsilon)})^k\big] = p_t^*(x,y)^2\mathbb{E}^X_{x,y;t}\mathbb{E}^{X'}_{x,y;t}\big[(L_t)^k\big],$$

and thus we have shown that

$$p_t^*(x,y)^2\mathbb{E}^X_{x,y;t}\mathbb{E}^{X'}_{x,y;t}\big[(L_t)^k\big] = k!\int_{\Delta_k(t)}\int_{\mathbb{R}^k_+} P_k^{t,x,y}(\mathbf{s},\mathbf{z})^2\prod_{i=1}^k \mathrm{d}z_i\mathrm{d}s_i.$$

Rearranging completes the proof. □

Lemma 2.2.8.

$$\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1}}\frac{1}{\sqrt{t-s_1}}\prod_{i=2}^k\frac{1}{\sqrt{s_{i-1}-s_i}}\frac{1}{\sqrt{s_k}}\prod_{i=1}^k \mathrm{d}s_i = \frac{\pi^{(k+1)/2}t^{(k-1)/2}}{\Gamma\big(\frac{k+1}{2}\big)}.$$

Proof. By the change of variables $s/t \mapsto u$ and the Euler Beta integral [OLBC10, equation 5.12.1]

$$\int_0^1 u^{a-1}(1-u)^{b-1}\,\mathrm{d}u = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\quad a,b>0, \tag{2.21}$$

we have

$$\int_0^t s^{a-1}(t-s)^{b-1}\,\mathrm{d}s = t^{a+b-1}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\quad a,b>0, \tag{2.22}$$

which implies that

$$\int_0^{s_{k-1}}\frac{1}{\sqrt{s_{k-1}-s_k}}\frac{1}{\sqrt{s_k}}\,\mathrm{d}s_k = \pi, \tag{2.23}$$

since $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$. We now claim that for all $k \geq 2$:

$$\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-2}}\frac{1}{\sqrt{t-s_1}}\prod_{i=2}^{k-1}\frac{1}{\sqrt{s_{i-1}-s_i}}\prod_{i=1}^{k-1}\mathrm{d}s_i = \frac{\pi^{(k-1)/2}t^{(k-1)/2}}{\Gamma\big(\frac{k+1}{2}\big)}. \tag{2.24}$$

The case $k = 2$ is true by a simple integration:

$$\int_0^t \frac{1}{\sqrt{t-s_1}} \mathrm{d}s_1 = 2\sqrt{t} = \frac{\sqrt{\pi t}}{\Gamma(3/2)},$$

since $\Gamma(3/2) = \sqrt{\pi}/2$. Assume the statement is true for $k-1$, then

$$\begin{aligned}\int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-2}} \frac{1}{\sqrt{t-s_1}} \prod_{i=2}^{k-1} \frac{1}{\sqrt{s_{i-1}-s_i}} \prod_{i=1}^{k-1} \mathrm{d}s_i &= \int_0^t \frac{1}{\sqrt{t-s_1}} \frac{\pi^{(k-2)/2} s_1^{(k-2)/2}}{\Gamma\left(\frac{k}{2}\right)} \mathrm{d}s_1 \\ &= \frac{(\pi t)^{(k-1)/2}}{\Gamma\left(\frac{k+1}{2}\right)},\end{aligned}$$

by the induction hypothesis and (2.22). Finally, combining (2.23) and (2.24) completes the proof. □

Lemma 2.2.9. *For all $a \geq 1$ and $T > 0$ there exists a constant $C := C(a,T) > 0$ such that for all $0 < t \leq T$*

$$\sup_{x,x',y,y' \in \mathbb{R}_+} \frac{p_t^*(x,y)}{xy} \frac{p_t^*(x',y')}{x'y'} \mathbb{E}^X_{x,y;t} \mathbb{E}^{X'}_{x',y';t} \left[\exp(aL_t)\right] < Ct^{-3}.$$

Proof. As in the proof of Lemma 2.2.6, we decompose $L_t = L_{[0,t/2]} + L_{[t/2,t]}$. We first estimate the first term. We will show that $L_{[0,t/2]}$ has exponential moments of all orders uniformly in x and x'. Recall that the law of the Bessel bridge $X_{t/2}$ is absolutely continuous with respect to the law of a Bessel process started from x with a Radon–Nikodym derivative equal to $q_{t/2}(X_{t/2},y)/q_t(x,y) = \frac{p^*_{t/2}(X_{t/2},y)}{X_{t/2}y} \frac{xy}{p_t^*(x,y)}$. By Lemma 2.2.2 this is bounded above by $c_{t/2} \frac{xy}{p_t^*(x,y)}$ where $c_t = \sqrt{2/\pi t^3}$ and so

$$\begin{aligned}\mathbb{E}^X_{x,y;t} \mathbb{E}^{X'}_{x',y';t} \left[\exp(aL_{[0,t/2]})\right] &\leq \mathbb{E}^{\mathrm{Bes}}_x \mathbb{E}^{\mathrm{Bes}}_{x'} \left[\frac{q_{t/2}(X_{t/2},y)}{q_t(x,y)} \frac{q_{t/2}(X'_{t/2},y')}{q_t(x',y')} \exp(aL_{[0,t/2]}) \right] \\ &\leq c_{t/2}^2 \frac{xy}{p_t^*(x,y)} \frac{x'y'}{p_t^*(x',y')} \mathbb{E}^{\mathrm{Bes}}_x \mathbb{E}^{\mathrm{Bes}}_{x'} \left[\exp(aL_{[0,t/2]})\right].\end{aligned}$$

We shall show that the latter expectation is bounded uniformly in x and x' for all a. It can be shown, for example using Itô's formula, that a three-dimensional Bessel process X started from x satisfies the stochastic differential equation,

$$X_t = x + \beta_t + \int_0^t \frac{1}{X_s} \, \mathrm{d}s,$$

where β_s is a standard Brownian motion, see also [RY99, Chapter VI.3]. Hence, by Tanaka's formula [RY99, Chapter VI, Theorem 1.2], we have

$$
\begin{aligned}
L_{[0,t/2]} &= |X_{t/2} - X'_{t/2}| - |x - x'| - \int_0^{t/2} \operatorname{sgn}(X_s - X'_s)\, \mathrm{d}(\beta_s - \beta'_s) \\
&\quad - \int_0^{t/2} \operatorname{sgn}(X_s - X'_s) \left(\frac{1}{X_s} - \frac{1}{X'_s} \right) \mathrm{d}s \\
&\le |X_{t/2} - x| + |X'_{t/2} - x'| + \left| \int_0^{t/2} \operatorname{sgn}(X_s - X'_s)\, \mathrm{d}(\beta_s - \beta'_s) \right| \\
&\quad + \int_0^{t/2} \frac{1}{X_s}\, \mathrm{d}s + \int_0^{t/2} \frac{1}{X'_s}\, \mathrm{d}s.
\end{aligned}
$$

We bound each term in the last line separately. Firstly, by Lévy's characterisation theorem, the third term above is a Brownian motion and therefore has exponential moments of all orders. We now estimate $|X_{t/2} - x|$. We make use of the fact that a three-dimensional Bessel process started at x can be realised as the positive eigenvalue of a 2×2 traceless Hermitian matrix defined by

$$
\mathbf{B}_t := \begin{pmatrix} B_t^1 & B_t^2 + iB_t^3 \\ B_t^2 - iB_t^3 & -B_t^1 \end{pmatrix},
$$

where B_t^i, $1 \le i \le 3$ are independent Brownian motions with $B_0^i = x_i$ and $x^2 = x_1^2 + x_2^2 + x_3^2$. Denote by $\lambda(\mathbf{B}_t)$ the positive eigenvalue of $\mathbf{B}_t$ then by Weyl's inequality [Bha97, Theorem III.2.1], $\lambda(\mathbf{B}_{t/2}) = \lambda(\mathbf{B}_{t/2} - \mathbf{B}_0 + \mathbf{B}_0) \le \lambda(\mathbf{B}_{t/2} - \mathbf{B}_0) + \lambda(\mathbf{B}_0)$ and so $|\lambda(\mathbf{B}_{t/2}) - \lambda(\mathbf{B}_0)| \le \lambda(\mathbf{B}_{t/2} - \mathbf{B}_0)$. Consequently, $|X_{t/2} - x|$ is bounded by $\lambda(\mathbf{B}_{t/2} - \mathbf{B}_0)$ which is independent of x. The latter is bounded by its matrix norm and since the entries of $B_{t/2} - B_0$ are independent standard Brownian motions, its matrix norm is therefore a product and sum of independent Gaussian random variables which has exponential moments of all orders. The same argument applies to $|X'_{t/2} - x'|$. As a result, $\int_0^{t/2} X_s^{-1}\, \mathrm{d}s = X_{t/2} - x - \beta_{t/2}$ also have exponential moments of all orders and likewise for $\int_0^{t/2} (X'_s)^{-1}\, \mathrm{d}s$. Therefore, the same is true for $L_{[0,t/2]}$ and moreover, the bound is independent of the starting points x and x'.

For the exponential moments of $L_{[t/2,t]}$, we can by time reversal consider $L_{[t/2,t]}$ as the local time of the difference between two Bessel bridges starting from y, y' and ending at x, x' at time t. The same argument as above applies to show that under $\mathbb{P}_y^{\mathrm{Bes}}$ and $\mathbb{P}_{y'}^{\mathrm{Bes}}$, $L_{[t/2,t]}$ has for each $t > 0$, exponential moments of all orders uniformly in y and y'.

Finally, by the Cauchy–Schwarz inequality and the above arguments we have for all a and $0 < t \le T$ there exists a $C := C(a, T) > 0$ such that for all x, x', y, y' (denoting

$\mathbb{E} = \mathbb{E}^X_{x,y;t}\mathbb{E}^{X'}_{x',y';t}$ for brevity)

$$\begin{aligned}
\mathbb{E}[e^{aL_t}] &\leq \sqrt{\mathbb{E}[e^{2aL_{[0,t/2]}}]\mathbb{E}[e^{2aL_{[t/2,t]}}]} \\
&= \left(c^2_{t/2}\frac{xy}{p^*_t(x,y)}\frac{x'y'}{p^*_t(x',y')}\mathbb{E}^{\mathrm{Bes}}_x\mathbb{E}^{\mathrm{Bes}}_{x'}[e^{2aL_{[0,t/2]}}]\right)^{1/2} \\
&\quad \times \left(c^2_{t/2}\frac{yx}{p^*_t(y,x)}\frac{y'x'}{p^*_t(y',x')}\mathbb{E}^{\mathrm{Bes}}_y\mathbb{E}^{\mathrm{Bes}}_{y'}[e^{2aL_{[t/2,t]}}]\right)^{1/2} \\
&\leq Cc^2_{t/2}\frac{xy}{p^*_t(x,y)}\frac{x'y'}{p^*_t(x',y')},
\end{aligned}$$

as required. □

2.2.3 Some Useful Results

We say that a random field f is in $\mathscr{P}_2$ if it is predictable and $\int_0^\infty\int_{\mathbb{R}}\|f(s,y)\|_2^2\,\mathrm{d}y\mathrm{d}s < \infty$. We say that f is $L^2(\Omega)$-continuous on a set $K \subseteq (0,\infty)\times\mathbb{R}$ if for all $(s,y)\in K$, $\lim_{(s',y')\to(s,y)}\|f(s',y') - f(s,y)\|_2 = 0$. Since the Walsh integral is defined for random fields in $\mathscr{P}_2$ (see Appendix A), it is convenient to have a set of conditions to verify the predictability of a random field. The following result is from [CD15a, Proposition 3.1] which is an extension of [DF98, Proposition 2] to space-time white noise.

Proposition 2.2.10. *Let $t > 0$ and suppose a random field $\big(f(s,y),(s,y)\in(0,t)\times\mathbb{R}\big)$ satisfies*

(i) f is adapted, that is for all $(s,y)\in(0,t)\times\mathbb{R}$, $f(s,y)$ is $\mathscr{F}_s$-measurable;

(ii) for all $(s,y)\in(0,t)\times\mathbb{R}$, $\|f(s,y)\|_2<\infty$ and $(s,y)\mapsto f(s,y)$ is $L^2(\Omega)$-continuous on $(0,t)\times\mathbb{R}$;

(iii) $\int_0^t\int_{\mathbb{R}}\|f(s,y)\|_2^2\,\mathrm{d}y\mathrm{d}s<\infty$.

Then $f\in\mathscr{P}_2$ and

$$\int_0^t\int_{\mathbb{R}} f(s,y)\,W(\mathrm{d}s,\mathrm{d}y),$$

is a well-defined Walsh integral.

The next result is a bound on the $L^p(\Omega)$ norm of stochastic integrals which will be used repeatedly. See for example [Kho14, Proposition 4.4], [CK12, Lemma 2.4] and [FK09, Lemma 3.3].

Lemma 2.2.11. *Define a random field $\big(f(t,y);(t,y)\in(0,\infty)\times\mathbb{R}\big)$ by*

$$f(t,y) = \int_0^t\int_{\mathbb{R}}\Gamma_{t-s}(y,z)w(s,z)\,W(\mathrm{d}s,\mathrm{d}z),$$

where w is a predictable random field and $\Gamma_t(y,z)$ is a non-random measurable function on $(0,\infty)\times\mathbb{R}^2$ such that $\int_0^t\int_{\mathbb{R}}\Gamma_{t-s}(y,z)^2\|w(s,z)\|_2^2\,\mathrm{d}z\mathrm{d}s<\infty$ for all $(t,y)\in(0,\infty)\times\mathbb{R}$. Then *for all integers $p \geq 2$, $t \geq 0$ and $y \in \mathbb{R}$*

$$\|f(t,y)\|_p^2 \leq c_p^2 \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(y,z)^2 \|w(s,z)\|_p^2 \, \mathrm{d}z\mathrm{d}s,$$

where $c_p \leq 2\sqrt{p}$ is the constant from the Burkholder–Davis–Gundy inequality.

We also need the following $L^p(\Omega)$ bound on multiple stochastic integrals.

Lemma 2.2.12. *For all $k \geq 1$ and deterministic $f \in L^2(\Delta_k(t) \times \mathbb{R}^k)$ we have*

$$\left\| \int_{\Delta_k(t)} \int_{\mathbb{R}^k} f(\mathbf{s},\mathbf{y}) \, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) \right\|_p^2 \leq c_p^{2k} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} f(\mathbf{s},\mathbf{y})^2 \, \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{s}.$$

Proof. Since multiple stochastic integrals on $\Delta_k(t)$ coincide with iterated stochastic integrals we can apply the Burkholder–Davis–Gundy inequality and Minkowski's integral inequality k times to obtain

$$\begin{aligned}
&\left\| \int_{\Delta_k(t)} \int_{\mathbb{R}^k} f(\mathbf{s},\mathbf{y}) \, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) \right\|_p^2 \\
&\qquad \leq c_p^2 \int_0^t \int_{\mathbb{R}} \left\| \int_0^{s_1} \cdots \int_0^{s_{k-1}} \int_{\mathbb{R}^{k-1}} f(\mathbf{s},\mathbf{y}) \, W(\mathrm{d}s_k, \mathrm{d}y_k) \ldots W(\mathrm{d}s_2, \mathrm{d}y_2) \right\|_p^2 \mathrm{d}y_1 \mathrm{d}s_1 \\
&\qquad \vdots \\
&\qquad \leq c_p^{2k} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} f(\mathbf{s},\mathbf{y})^2 \, \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{s}.
\end{aligned}$$

□

2.3 Existence, Uniqueness and Moment Estimates

2.3.1 Bounded Initial Data

We now prove the existence, uniqueness and moment estimates part of Theorem 2.1.1(a). The proof of continuity will be delayed to Section 2.4. In the sequel constants will generally be denoted by c, C or K and possibly adorned with primes, or subscripts. They may differ from line to line and their dependence if any will always be specified. However, C_i, $1 \leq i \leq 4$ will always mean the constants in Lemma 2.2.1 and Proposition 2.2.4. $T \geq 0$ will always denote the finite time horizon.

Proof of existence, uniqueness and moment estimates of Theorem 2.1.1(a). The proof is by a Picard iteration argument. Throughout the proof, we fix an arbitrary integer $p \geq 2$. For

$(t,y) \in (0,\infty) \times \mathbb{R}_+$ define $v^0(t,y) := J(t,y)$ where J was defined in (2.1) and for $n \geq 1$, let

$$\begin{aligned} v^n(t,y) &= v^0(t,y) + \int_0^t \int_0^\infty q_{t-s}(y,z) v^{n-1}(s,z)\, W(\mathrm{d}s, \mathrm{d}z) \\ &=: v^0(t,y) + I^n(t,y). \end{aligned}$$

We first show that each of the stochastic integrals above are well defined, that is, for all n and $(t,y) \in (0,\infty) \times \mathbb{R}_+$, $q_{t-\cdot}(y,\cdot)v^n(\cdot,\cdot) \in \mathscr{P}_2$ i.e., $q_{t-\cdot}(y,\cdot)v^n(\cdot,\cdot)$ is a predictable random field such that $\int_0^t \int_0^\infty q_{t-s}(y,z)^2 \|v^n(s,z)\|_2^2 \,\mathrm{d}z\mathrm{d}s < \infty$.

Since the initial data g is $\mathscr{F}_0$-measurable, v^0 is adapted to the filtration $\big(\mathscr{F}_t, t \geq 0\big)$. Now by Lemma 2.4.2, $(t,y) \mapsto J(t,y)$ is $L^2(\Omega)$-continuous on $(0,\infty) \times \mathbb{R}_+$ and for each (t,y), $q_{t-\cdot}(y,\cdot)$ is determinstic and continuous on $[0,t) \times \mathbb{R}_+$, hence the random field $q_{t-\cdot}(y,\cdot)v^0(\cdot,\cdot)$ is adapted and $L^2(\Omega)$-continuous on $(0,t) \times \mathbb{R}_+$. By the assumptions on g, $\sup_{y\in\mathbb{R}_+} \|g(y)\|_p \leq K_{p,g} < \infty$ and so by Minkowski's integral inequality for all $t > 0$

$$\|v^0(t,y)\|_p \leq \sup_{z\in\mathbb{R}_+} \|g(z)\|_p \int_0^\infty q_t(y,z)\, \mathrm{d}z = \sup_{z\in\mathbb{R}_+} \|g(z)\|_p \leq K_{p,g}, \tag{2.25}$$

and so $\sup_{(t,y)\in\mathbb{R}_+\times\mathbb{R}_+} \|v^0(t,y)\|_p^2 \leq K_{p,g}^2$. This and Lemma 2.2.1 imply that

$$\int_0^t \int_0^\infty q_{t-s}(y,z)^2 \|v^0(s,z)\|_p^2 \,\mathrm{d}z\mathrm{d}s \leq C_2 K_{p,g}^2 t^{1/2}. \tag{2.26}$$

Hence, by Proposition 2.2.10, $q_{t-\cdot}(y,\cdot)v^0(\cdot,\cdot) \in \mathscr{P}_2$ for all $(t,y) \in (0,\infty) \times \mathbb{R}_+$ and so $I^1(t,y)$ is a well-defined Walsh integral. Consequently, the random field $\big(v^1(t,y) = v^0(t,y) + I^1(t,y), (t,y) \in (0,\infty) \times \mathbb{R}_+\big)$ is well defined.

We shall show below that the sequence $\{v^k(t,y)\}_{k\geq0}$ is Cauchy in $L^p(\Omega)$ and for this we define $d_k(t,y) := \|v^{k+1}(t,y) - v^k(t,y)\|_p$. By Lemma 2.2.11 and (2.26), we have for all $(t,y) \in (0,\infty) \times \mathbb{R}_+$,

$$\begin{aligned} d_0(t,y)^2 &\leq c_p^2 \int_0^t \int_0^\infty q_{t-s}(y,z)^2 \|v^0(s,z)\|_p^2 \,\mathrm{d}z\mathrm{d}s \\ &\leq 2C_1 c_p^2 K_{p,g}^2 \sqrt{t} \\ &\leq C_1 c_p^2 K_{p,g}^2 \sqrt{\pi} \frac{\sqrt{t}}{\Gamma(3/2)}, \end{aligned}$$

since $\Gamma(3/2) = \sqrt{\pi}/2$.

Now assume that for all $0 \leq k \leq n$, $\big(v^k(t,y), (t,y) \in (0,\infty) \times \mathbb{R}_+\big)$ is well defined and satisfies

(i) v^k is adapted to the filtration $\big(\mathscr{F}_t, t \geq 0\big)$,

(ii) $(t,y) \mapsto v^k(t,y)$ is $L^2(\Omega)$-continuous on $(0,\infty) \times \mathbb{R}_+$,

(iii) for all $(t,y) \in (0,\infty) \times \mathbb{R}_+$ and $0 \le k \le n-1$

$$d_k(t,y)^2 \le K_{p,g}^2 (C_1 c_p^2 \sqrt{\pi})^{k+1} \frac{t^{(k+1)/2}}{\Gamma\big(\frac{k+1}{2}+1\big)}.$$

We want to show that the same is true for v^{n+1} and d_n. Let $(t,y) \in (0,\infty) \times \mathbb{R}_+$. Clearly, $v^n(t,y) = v^0(t,y) + \sum_{k=1}^n v^k(t,y) - v^{k-1}(t,y)$ and so to bound the moments of v^k it suffices to bound each of the d_k's. Indeed, by property (iii) and the bound (2.25) we have

$$\|v^n(t,y)\|_p^2 \le 2\|v^0(t,y)\|_p^2 + \sum_{k=1}^n 2^k d_{k-1}(t,y)^2 \le 2K_{p,g}^2 \sum_{k=0}^n (2C_1 c_p^2 \sqrt{\pi})^k \frac{t^{k/2}}{\Gamma\big(\frac{k}{2}+1\big)}, \quad (2.27)$$

which shows that for all $n \ge 1$ and $T > 0$, $\sup_{(t,y)\in[0,T]\times\mathbb{R}_+} \|v^n(t,y)\|_p^2 < \infty$. Using the above bound, we have by Lemma 2.2.1 and equation (2.22) that

$$\begin{aligned}
&\int_0^t \int_0^\infty q_{t-s}(y,z)^2 \|v^n(s,z)\|_p^2 \,\mathrm{d}z\mathrm{d}s \\
&\qquad \le 2K_{p,g}^2 \sum_{k=0}^n (2C_1 c_p^2 \sqrt{\pi})^k \int_0^t \frac{s^{k/2}}{\Gamma\big(\frac{k}{2}+1\big)} \int_0^\infty q_{t-s}(y,z)^2 \,\mathrm{d}z\mathrm{d}s \\
&\qquad \le 2K_{p,g}^2 \sum_{k=0}^n (2C_1 c_p^2 \sqrt{\pi})^k C_1 \sqrt{\pi} \frac{t^{(k+1)/2}}{\Gamma\big(\frac{k+1}{2}+1\big)} \\
&\qquad < \infty. \qquad (2.28)
\end{aligned}$$

This and the induction hypothesis, the fact that $q_{t-\cdot}(y,\cdot)$ is deterministic and continuous shows that $q_{t-\cdot}(y,\cdot)v^n(\cdot,\cdot) \in \mathscr{P}_2$ for all $(t,y) \in (0,\infty)\times\mathbb{R}_+$ and hence the stochastic integral I^{n+1} is well defined in the sense of Walsh. Moreover, it is adapted and so $v^{n+1} = v^0 + I^{n+1}$ is also adapted. By Proposition 2.2.4, the isometry property of the stochastic integral and the moment bound (2.27), we see that for all $0 < t \le t' < \infty$ and y, $y' \in \mathbb{R}_+$

$$\begin{aligned}
&\|I^{n+1}(t,y) - I^{n+1}(t',y')\|_2^2 \\
&\qquad \le 2\int_0^t \int_0^\infty \|v^n(s,z)\|_2^2 \big(q_{t-s}(y,z) - q_{t'-s}(y',z)\big)^2 \,\mathrm{d}z\mathrm{d}s \\
&\qquad\qquad + 2\int_t^{t'} \int_0^\infty \|v^n(s,z)\|_2^2 q_{t'-s}(y',z)^2 \,\mathrm{d}z\mathrm{d}s \\
&\qquad \le 2 \sup_{(s,z)\in[0,t']\times\mathbb{R}_+} \|v^n(s,z)\|_2^2 \max(C_2,C_3,C_4)\big(|y-y'| + |t-t'|^{\frac{1}{2}}\big),
\end{aligned}$$

which proves the $L^2(\Omega)$-continuity of $(t,y) \mapsto I^{n+1}(t,y)$ on $(0,\infty)\times\mathbb{R}_+$. Consequently, $v^{n+1} = v^0 + I^{n+1}$ is also $L^2(\Omega)$-continuous.

For the bound on d_n, we use the recursive property of $\{d_n\}_{n\ge0}$, property (iii) and

Lemma 2.2.1 to obtain

$$
\begin{aligned}
d_n(t,y)^2 &\le c_p^2 \int_0^t \int_0^\infty q_{t-s}(y,z)^2 d_{n-1}(s,z)^2 \,\mathrm{d}z\mathrm{d}s \\
&\le K_{p,g}^2 (C_1 c_p^2)^{n+1} \pi^{n/2} \int_0^t \frac{s^{n/2}}{\Gamma\big(\frac{n}{2}+1\big)} (t-s)^{-1/2} \,\mathrm{d}s \\
&= K_{p,g}^2 (C_1 c_p^2 \sqrt{\pi})^{n+1} \frac{t^{(n+1)/2}}{\Gamma\big(\frac{n+1}{2}+1\big)},
\end{aligned} \tag{2.29}
$$

where we have used (2.22) in the last line. It follows that the bound (2.27) holds with n replaced with $n+1$ and therefore the same is true for (2.28). Hence, v^{n+1} satisfies all the assumptions of Proposition 2.2.10 and therefore $v^{n+1} \in \mathscr{P}_2$. We conclude that for all integers $n \ge 0$ the random field $\big(v^n(t,y) = v^0(t,y) + I^n(t,y), (t,y) \in (0,\infty) \times \mathbb{R}_+\big)$ is well defined and satisfies properties (i), (ii) and (iii) listed above.

We now show that for each (t,y) the sequence $\{v^n(t,y)\}_{n\ge 0}$ is Cauchy in $L^p(\Omega)$. This follows from the fact that for all $T > 0$

$$
\sup_{(t,y)\in[0,T]\times\mathbb{R}_+} \sum_{n=0}^{\infty} d_n(t,y) < \infty.
$$

Indeed,

$$
\frac{\Gamma\big(\frac{n}{2}+1\big)}{\Gamma\big(\frac{n+1}{2}+1\big)} \approx \Big(\frac{n}{2}\Big)^{-1/2} \to 0 \quad \text{as } n \to \infty,
$$

where we have used the following asymptotic of Gamma functions: for any a, $b \in \mathbb{R}$, $\frac{\Gamma(z+a)}{\Gamma(z+b)} \approx z^{a-b}$ as $z \to \infty$, see [OLBC10, equation 5.11.12]. The result then follows from property (iii) and the ratio test. We conclude that there exists a random field $v(t,y)$ such that $v^n(t,y) \to v(t,y)$ in $L^p(\Omega)$ and almost surely for a subsequence uniformly for $(t,y) \in [0,T] \times \mathbb{R}_+$.

Since each v^n is adapted v is also adapted. The $L^2(\Omega)$-continuity of v is inherited from that of v^n since the convergence is uniform on $[0,T] \times \mathbb{R}_+$ for all $T > 0$. Taking limit as $n \to \infty$ on both sides of (2.27) and using the uniform $L^p(\Omega)$ convergence of $v^n(t,y)$ and noting that $2C_1\sqrt{\pi} = 3/2$ we have

$$
\|v(t,y)\|_p^2 \le 2K_{p,g}^2 \sum_{k=1}^{\infty} \left(\frac{3c_p^2}{2}\right)^{k-1} \frac{t^{(k-1)/2}}{\Gamma\left(\frac{k+1}{2}\right)}.
$$

By Proposition 2.2 of [CD15a], we know that for all $x \ge 0$

$$
e^{x^2}\big(1 + \operatorname{erf}(x)\big) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{\Gamma\left(\frac{k+1}{2}\right)}. \tag{2.30}
$$

Using this with $x = 3c_p^2 t^{1/2}/2$ shows that

$$
\|v(t,y)\|_p^2 \le 2K_{p,g}^2 e^{9c_p^4 t/4}\big(1 + \operatorname{erf}(3c_p^2 t^{1/2}/2)\big).
$$

This proves the moment bound (2.7). This bound and Lemma 2.2.1 implies that

$$\int_0^t \int_0^\infty q_{t-s}(y,z)\|v(s,z)\|_2^2 \, \mathrm{d}z\mathrm{d}s < \infty,$$

and so by Proposition 2.2.10, for all $(t,y) \in (0,\infty) \times \mathbb{R}_+$, the random field $q_{t-\cdot}(y,\cdot)v(\cdot,\cdot) \in \mathscr{P}_2$ and

$$I(t,y) = \int_0^t \int_0^\infty q_{t-s}(y,z)v(s,z)\, W(\mathrm{d}s,\mathrm{d}z),$$

is a well-defined Walsh integral.

It remains to show that the limit $v(t,y)$ solves (2.1). Fix $(t,y) \in [0,\infty) \times \mathbb{R}_+$. By definition, $v^n(t,y) = v^0(t,y) + I^n(t,y)$ where the left hand side converges in $L^p(\Omega)$ and almost surely for a subsequence to $v(t,y)$. On the other hand,

$$\begin{aligned}
\|I^n(t,y) - I(t,y)\|_p^2 &\le c_p^2 \int_0^t \int_0^\infty q_{t-s}(y,z)^2 \|v^{n-1}(s,z) - v(s,z)\|_p^2 \, \mathrm{d}z\mathrm{d}s \\
&\le c_p^2 \sup_{(s,z)\in[0,t]\times\mathbb{R}_+} \|v^{n-1}(s,z) - v(s,z)\|_p^2 \int_0^t \int_0^\infty q_{t-s}(y,z)^2 \, \mathrm{d}z\mathrm{d}s \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore, $I^n(t,y)$ converges in $L^p(\Omega)$ to $I(t,y)$ and hence converges almost surely for a subsequence to the same limit. The limit of both sides of $v^n(t,y) = v^0(t,y) + I^n(t,y)$ must be equal almost surely and thus we have that shown that v is a predictable random field which satisfies (2.1) almost surely for all $(t,y) \in [0,\infty) \times \mathbb{R}_+$ as required. This proves existence.

For uniqueness, suppose v_1 and v_2 are two solutions satisfying (2.1). Let $d(t,y) = \|v_1(t,y) - v_2(t,y)\|_p$ then by a similar calculation as for existence we have

$$d(t,y)^2 \le \sup_{(s,y)\in[0,t]\times\mathbb{R}_+} d(s,y)^2 \left(\frac{3c_p^2}{2}\right)^n \frac{t^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)},$$

which converges to 0 as $n \to \infty$. Therefore $d \equiv 0$ and so for all (t,y), $v_1(t,y) = v_2(t,y)$ almost surely i.e. v_1 and v_2 are versions of each other. This completes the proof of existence, uniqueness and moment estimates for bounded initial data. □

2.3.2 Delta Initial Data

We now prove the existence, uniqueness and moment estimates part of Theorem 2.1.1(b). Recall that in this case the mild equation reads

$$\begin{aligned}
v(t,x,y) &= \frac{p_t^*(x,y)}{xy} + \int_0^t \int_0^\infty q_{t-s}(y,z)v(s,x,z)\, W(\mathrm{d}s,\mathrm{d}y) \\
&=: J(t,x,y) + I(t,x,y).
\end{aligned}$$

In the previous section we have shown that the sequence of approximating solutions $\{v^n\}_{n\geq 0}$ is Cauchy in $L^p(\Omega)$ by showing that the sum $\sum_{n=0}^{\infty}\|v^{n+1}-v^n\|_p$ is finite. Central to the estimate of the sum is that $\|v^0(t,y)\|_p$ is bounded uniformly in space and time. In the present case, the methods of the previous section fail because $J(t,x,y)$ is not bounded uniformly in time for all x, y since

$$\frac{p_t^*(x,x)}{x^2} = \frac{1}{x^2\sqrt{2\pi t}}(1-e^{-2x^2/t}),$$

whose supremum over $t\in[0,\infty)$ is infinite. Instead, we shall propose a solution to the above integral equation as a chaos series, show that the series is well defined by writing its $L^2(\Omega)$ norm in terms of local times at 0 of the difference of two independent Bessel bridges using Proposition 2.2.7 and then show that this chaos series satisfies the mild equation.

Proof of Theorem 2.1.1(b). Throughout the proof we fix an arbitrary integer $p\geq 2$. We first prove uniqueness. Suppose $v_1(t,x,y)$ and $v_2(t,x,y)$ are two solutions to (2.3) then $V(t,x,y):=v_1(t,x,y)-v_2(t,x,y)$ by linearity of the equation is the solution of (2.1) with initial condition $g\equiv 0$, i.e., it satisfies the mild form

$$V(t,x,y)=\int_0^t\int_0^\infty q_{t-s}(y,z)V(s,x,z)\,W(\mathrm{d}s,\mathrm{d}z).$$

Let $d(t,x,y)=\|V(t,x,y)\|_p$ then by the same argument as in the proof of uniqueness in part (a) of the theorem we have $d(t,x,y)\equiv 0$ which implies that $v_1(t,x,y)=v_2(t,x,y)$ almost surely for all (t,x,y), hence uniqueness.

Now define a random field for $\mathbf{s}=(s_1,\dots,s_k)\in\Delta_k(t)$, $\mathbf{z}=(z_1,\dots,z_k)\in\mathbb{R}_+^k$, $t>0$ and x, $y\in\mathbb{R}_+$ by

$$v(t,x,y)=J(t,x,y)\bigg(1+\sum_{k=1}^{\infty}\int_{\Delta_k(t)}\int_{\mathbb{R}_+^k}R_k(\mathbf{s},\mathbf{z};t,x,y)\,W^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{z})\bigg),\tag{2.31}$$

where R_k was defined in Proposition 2.2.7. By the isometry property (A.6) of multiple stochastic integrals and Proposition 2.2.7, we have

$$\begin{aligned}\|v(t,x,y)\|_2^2&=J(t,x,y)^2\bigg(1+\sum_{k=1}^{\infty}\int_{\Delta_k(t)}\int_{\mathbb{R}_+^k}R_k(\mathbf{s},\mathbf{z};t,x,y)^2\prod_{i=1}^{k}\mathrm{d}z_i\mathrm{d}s_i\bigg)\\&=J(t,x,y)^2\mathbb{E}_{x,y;t}^{X,X'}\big[\exp(L_t)\big],\end{aligned}$$

where $\mathbb{E}_{x,y;t}^{X,X'}$ denotes the expectation with respect to the law of two independent copies of a Bessel bridge starting from x at time 0 and ending in y at time t and $L_t=L_t(X-X')$ is the local time at 0 of the difference of such processes. By Lemma 2.2.9, the above is finite hence the chaos series (2.31) converges in $L^2(\Omega)$ and v is well defined. We shall show that v satisfies equation (2.3) almost surely for all $(t,x,y)\in(0,\infty)\times\mathbb{R}_+\times\mathbb{R}_+$.

Let $v^0(t,x,y) = J(t,x,y)$ and for $n \geq 1$ let

$$v^n(t,x,y) = v^0(t,x,y)\left(1 + \sum_{k=1}^{n} \int_{\Delta_k(t)} \int_{\mathbb{R}_+^k} R_k(\mathbf{s},\mathbf{z};t,x,y)\, W^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{z})\right).$$

In other words, v^n is the nth partial sum of v. The random field v^n satisfies the integral equation

$$v^n(t,x,y) = v^0(t,x,y) + I^n(t,x,y), \tag{2.32}$$

where

$$I^n(t,x,y) = \int_0^t \int_0^\infty q_{t-s}(y,z) v^{n-1}(s,x,z)\, W(\mathrm{d}s,\mathrm{d}z).$$

Indeed, let $\mathbf{s}_{2:k+1} = (s_2,\ldots,s_{k+1}) \in \Delta_k(s_1)$ and $\mathbf{z}_{2:k+1} = (z_2,\ldots,z_{k+1}) \in \mathbb{R}_+^k$ then by the definition of v^{n-1}

$$\begin{aligned}
&\int_0^t \int_0^\infty q_{t-s_1}(y,z_1) v^{n-1}(s_1,x,z_1)\, W(\mathrm{d}s_1,\mathrm{d}z_1) \\
&\quad= \int_0^t \int_0^\infty q_{t-s_1}(y,z_1) v^0(s_1,x,z_1)\, W(\mathrm{d}s_1,\mathrm{d}z_1) \\
&\qquad + \sum_{k=1}^{n-1} \int_0^t \int_{\Delta_k(s_1)} \int_{\mathbb{R}_+^{k+1}} q_{t-s_1}(y,z_1) v^0(s_1,x,z_1) \\
&\qquad\qquad \times R_k(\mathbf{s}_{2:k+1},\mathbf{z}_{2:k+1};s_1,x,z_1)\, W^{\otimes k}(\mathrm{d}\mathbf{s}_{2:k+1},\mathrm{d}\mathbf{z}_{2:k+1}) W(\mathrm{d}s_1,\mathrm{d}z_1) \\
&\quad= v^0(t,x,y) \int_0^t \int_0^\infty R_1(s_1,z_1;t,x,y)\, W(\mathrm{d}s_1,\mathrm{d}z_1) \\
&\qquad + v^0(t,x,y) \sum_{k=1}^{n-1} \int_{\Delta_{k+1}(t)} \int_{\mathbb{R}_+^{k+1}} R_{k+1}(\mathbf{s},\mathbf{z};t,x,y)\, W^{\otimes k+1}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{z}) \\
&\quad= v^0(t,x,y) \sum_{k=1}^{n} \int_{\Delta_k(t)} \int_{\mathbb{R}_+^k} R_k(\mathbf{s},\mathbf{z};t,x,y)\, W^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{z}) \\
&\quad= v^n(t,x,y) - v^0(t,x,y).
\end{aligned}$$

Let $n \to \infty$ in equation (2.32) then the left hand side converges in $L^2(\Omega)$ to $v(t,x,y)$ whilst the right hand side converges in $L^2(\Omega)$ to $v^0(t,x,y) + I(t,x,y)$ since

$$\|I^n(t,x,y) - I(t,x,y)\|_2^2 = \int_0^t \int_0^\infty q_{t-s}(y,z)^2 \|v^{n-1}(s,x,z) - v(s,x,z)\|_2^2 \,\mathrm{d}z\mathrm{d}s,$$

which converges to zero by the $L^2(\Omega)$ convergence of v^{n-1} and the dominated convergence theorem. Indeed, by Proposition 2.2.7 and the monotone convergence theorem

$$\begin{aligned}\int_0^t \int_0^\infty & q_{t-s}(y,z)^2 \|v^{n-1}(s,x,z) - v(s,x,z)\|_2^2 \,\mathrm{d}z\mathrm{d}s \\ &\le 4 \int_0^t \int_0^\infty q_{t-s}(y,z)^2 v^0(s,x,z)^2 \mathbb{E}_{x,z;s}^{X,X'}\big[\exp(L_s)\big] \,\mathrm{d}z\mathrm{d}s \\ &= 4 \sum_{k=0}^\infty \frac{1}{k!} \int_0^t \int_0^\infty q_{t-s}(y,z)^2 v^0(s,x,z)^2 \mathbb{E}_{x,z;s}^{X,X'}\big[(L_s)^k\big] \,\mathrm{d}z\mathrm{d}s \\ &= 4 v^0(t,x,y)^2 \Big(\mathbb{E}_{x,y;t}^{X,X'}\big[\exp(L_t)\big] - 1\Big),\end{aligned}$$

which is finite by Lemma 2.2.9. We conclude that $v(t,x,y) = v^0(t,x,y) + I(t,x,y)$ almost surely for all $(t,x,y) \in (0,\infty) \times \mathbb{R}_+ \times \mathbb{R}_+$ which proves existence.

It remains to bound the pth moments of v. By Lemma 2.2.12, we have for each $k \ge 1$

$$\Big\| \int_{\Delta_k(t)} \int_{\mathbb{R}_+^k} R_k(\mathbf{s},\mathbf{z};t,x,y)\, W^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{z}) \Big\|_p^2 \le c_p^{2k} \int_{\Delta_k(t)} \int_{\mathbb{R}_+^k} R_k(\mathbf{s},\mathbf{z};t,x,y)^2 \,\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{s},$$

and so

$$\begin{aligned}\|v^n(t,x,y)\|_p^2 &\le 2v^0(t,x,y)^2 + v^0(t,x,y)^2 \sum_{k=1}^n 2^k c_p^{2k} \int_{\Delta_k(t)} \int_{\mathbb{R}_+^k} R_k(\mathbf{s},\mathbf{z};t,x,y)^2 \,\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{s} \\ &\le 2v^0(t,x,y)^2 \sum_{k=0}^n \frac{1}{k!} \mathbb{E}_{x,y;t}^{X,X'}\big[(2c_p^2 L_t)^k\big].\end{aligned}$$

Letting $n \to \infty$ we see that

$$\lim_{n\to\infty} \|v^n(t,x,y)\|_p^2 \le 2v^0(t,x,y)^2 \mathbb{E}_{x,y;t}^{X,X'}\big[\exp(2c_p^2 L_t)\big]. \tag{2.33}$$

By Cauchy–Schwarz inequality for any n, $m \ge 0$

$$\|v^n(t,x,y) - v^m(t,x,y)\|_p^p \le \|v^n(t,x,y) - v^m(t,x,y)\|_2 \|v^n(t,x,y) - v^m(t,x,y)\|_{2(p-1)}^{p-1},$$

which converges to zero as n, $m \to \infty$ and hence $v^n(t,x,y)$ converges to $v(t,x,y)$ in $L^p(\Omega)$ also for all $(t,x,y) \in (0,\infty) \times \mathbb{R}_+ \times \mathbb{R}_+$. Thus, we can replace the limit on the left hand side of (2.33) with $\|v(t,x,y)\|_p$ which gives the desired bound. This completes the proof of existence, uniqueness and moment estimates. □

2.4 Continuity

We shall use the following version of Kolmogorov's continuity criterion which is due to Chen and Dalang, see [CD14, Proposition 4.2].

Theorem 2.4.1. *Consider a random field* $\big(f(t,y) : (t,y) \in \mathbb{R}_+ \times \mathbb{R}^d\big)$. *Suppose there are* *constants $\alpha_0, \dots, \alpha_d \in (0,1]$ such that for all $p > 2(d+1)$ and all $M > 1$, there exist a constant $C := C(p,M)$ depending on p and M such that*

$$\|f(t,y) - f(s,x)\|_p \leq C\bigg(|t-s|^{\alpha_0} + \sum_{i=1}^{d} |y_i - x_i|^{\alpha_i}\bigg),$$

for all (t,y) and (s,x) in $[1/M, M] \times [-M, M]^d$. Then f has a modification which is locally Hölder continuous on $(0,\infty) \times \mathbb{R}^d$ with indices $(\beta\alpha_0, \dots, \beta\alpha_d)$ for all $\beta \in (0,1)$.

2.4.1 Bounded Initial Data

Recall that the solution satisfies the mild form $v(t,y) = J(t,y) + I(t,y)$ defined in equation (2.1) and the initial data g is assumed to satisfy $\sup_{z\in\mathbb{R}_+} \|g(z)\|_p \leq K_{p,g} < \infty$ for all $p \geq 2$. We shall show that each term on the right hand side satisfies the hypothesis of Theorem 2.4.1.

Lemma 2.4.2. *Let $M > 1$ and $p \geq 2$. There exist a constant $K := K(g, M, p) > 0$ such that for all y, $y' \in \mathbb{R}_+$ and t, $t' \in [1/M, M]$*

$$\|J(t,y) - J(t',y')\|_p \leq K\big(|y-y'| + |t-t'|\big).$$

Proof. Firstly, by Minkowski's integral inequality we have

$$\|J(t,y) - J(t',y')\|_p \leq K_{p,g} \int_0^\infty |q_t(y,z) - q_{t'}(y',z)| \, \mathrm{d}z.$$

We consider the spatial and the temporal increment separately.

By Lemma 2.2.3, we know that there is a $C > 0$ such that $\frac{\partial q_t}{\partial y}(y,z) \leq Ct^{-1/2} q_{2t}(y,z)$ and hence for all $t \in [1/M, M]$ and y, $y' \in \mathbb{R}_+$

$$\begin{aligned} \int_0^\infty |q_t(y,z) - q_t(y',z)| \, \mathrm{d}z &= \int_0^\infty \left| \int_y^{y'} \frac{\partial q_t}{\partial y}(y,z) \, \mathrm{d}y \right| \mathrm{d}z \\ &\leq \int_y^{y'} \int_0^\infty Ct^{-1/2} q_{2t}(y,z) \, \mathrm{d}z\mathrm{d}y \\ &\leq CM^{1/2}|y-y'|, \end{aligned}$$

where we have used the fact that q_t integrates to 1.

We now estimate the temporal increment. By Lemma 2.2.3 we know that $\frac{\partial q_t}{\partial t}(y,z) \leq Ct^{-1} q_{2t}(y,z)$ for some constant $C > 0$ and so in the same way as for the spatial increment we have for all t, $t' \in [1/M, M]$ and $y \in \mathbb{R}_+$ that

$$\int_0^\infty |q_t(y,z) - q_{t'}(y,z)| \, \mathrm{d}z \leq CM|t-t'|.$$

This completes the proof of the lemma. □

Lemma 2.4.3. *Let $M > 1$ and $p \geq 2$. There exists a constant $K := K(g, M, p) > 0$ such that for all (t, y) and $(t', y') \in [0, M] \times \mathbb{R}_+$*

$$\|I(t, y) - I(t', y')\|_p \leq K\big(|y - y'|^{1/2} + |t - t'|^{1/4}\big).$$

Proof. We consider the spatial and temporal increments separately. By (2.7) and the fact that $|\mathrm{erf}(\cdot)| \leq 1$, there is a finite constant $C := C(g, M, p) > 0$ such that

$$\sup_{(s,z)\in[0,t]\times\mathbb{R}_+} \|v(s, z)\|_p^2 \leq \sup_{(s,z)\in[0,M]\times\mathbb{R}_+} \|v(s, z)\|_p^2 \leq C.$$

Then in the same way as in the proof of $L^2(\Omega)$-continuity for bounded initial data we have by Lemma 2.2.11 and Proposition 2.2.4 that

$$\begin{aligned}\|I(t, y) - I(t, y')\|_p^2 &\leq c_p^2 \int_0^t \int_0^\infty \|v(s, z)\|_p^2 \big(q_{t-s}(y, z) - q_{t-s}(y', z)\big)^2 \,\mathrm{d}z\mathrm{d}s \\ &\leq C C_3 c_p^2 |y - y'|.\end{aligned}$$

For the temporal increment, assume without loss of generality that $0 \leq t' < t \leq M$ then by Proposition 2.2.4

$$\begin{aligned}\|I(t, y) - I(t', y)\|_p^2 \leq\; & 2c_p^2 \int_0^{t'} \int_0^\infty \|v(s, z)\|_p^2 \big(q_{t-s}(y, z) - q_{t'-s}(y, z)\big)^2 \,\mathrm{d}z\mathrm{d}s \\ &+ 2c_p^2 \int_{t'}^t \int_0^\infty q_{t-s}(y, z)^2 \|v(s, z)\|_p^2 \,\mathrm{d}z\mathrm{d}s \\ \leq\; & 2C c_p^2 \max(C_2, C_4)|t - t'|^{1/2},\end{aligned}$$

which completes the proof. □

Lemmata 2.4.2 and 2.4.3 together imply that for all $M > 1$ and $p \geq 2$ there is a constant C depending on p, M and the initial condition g such that

$$\|v(t, y) - v(t', y')\|_p \leq C\big(|t - t'|^{1/4} + |y - y'|^{1/2}\big),$$

for all (t, y) and $(t', y') \in [1/M, M] \times [0, M]$. Applying Kolmogorov's continuity criterion (Theorem 2.4.1) with $n = 2$, $\alpha_0 = 1/4$ and $\alpha_1 = 1/2$ shows that $(t, y) \mapsto v(t, y)$ is locally Hölder continuous over $(0, \infty) \times \mathbb{R}_+$ with indices $\alpha < 1/2$ in space and $\alpha < 1/4$ in time. This completes the proof of Theorem 2.1.1(a).

2.4.2 Delta Initial Delta

Let $v(t,x,y)$ be the solution to (2.3). In this case we cannot apply the methods used in the previous section directly since the moments of $v(t,x,y)$ blow up as $t \downarrow 0$. However, by the bound (2.8) and Lemma 2.2.9 we do have that for every $T > 0$ and $0 < t \leq T$ fixed that $\sup_{x,y\in\mathbb{R}_+} \|v(t,x,y)\|_p^2 \leq Ct^{-3}$ for a constant $C > 0$ depending on p and T. Thus, for all strictly positive times $v(t,x,y)$ belongs to the class of initial data in Theorem 2.1.1(a). It is clear that at any given time we can restart the equation taking the current solution as the new initial data. More precisely, let $\tau > 0$ and consider the shifted white noise $\dot{W}^\tau(s,y) = \dot{W}(\tau+s,y)$. Define $v^\tau(t,x,y) := v(\tau+t,x,y)$ then we have the following

Lemma 2.4.4. *For all $(t,x,y) \in (0,\infty)\times\mathbb{R}_+\times\mathbb{R}_+$, v^τ satisfies the following mild equation*

$$v^\tau(t,x,y) = \int_0^\infty v(\tau,x,z)q_t(y,z)\,\mathrm{d}z + \int_0^t\int_0^\infty q_{t-s}(y,z)v^\tau(s,x,z)\,W^\tau(\mathrm{d}s,\mathrm{d}z).$$

In other words, v^τ is the solution to (2.1) driven by the noise $\dot{W}^\tau$ with initial data $v(\tau,\cdot,\cdot)$.

Proof. By definition, $v(\tau+t,x,y)$ is the solution to (2.3) at time $\tau+t$. Denote the deterministic term and the stochastic integral term of (2.3) at time $\tau+t$ by $J^\tau(t,x,y)$ and $I^\tau(t,x,y)$ respectively, then by the Chapman–Kolmogorov equation for q_t we have

$$J^\tau(t,x,y) = \frac{p^*_{\tau+t}(x,y)}{xy} = \int_0^\infty \frac{p^*_\tau(x,y')}{xy'}q_t(y,y')\,\mathrm{d}y'.$$

On the other hand

$$\begin{aligned} I^\tau(t,x,y) &= \int_0^\tau\int_0^\infty + \int_\tau^{\tau+t}\int_0^\infty q_{\tau+t-s}(y,z)v(s,x,z)\,W(\mathrm{d}s,\mathrm{d}z) \\ &= \int_0^\tau\int_0^\infty\int_0^\infty q_t(y,y')q_{\tau-s}(y',z)\,\mathrm{d}y'\,v(s,x,z)\,W(\mathrm{d}s,\mathrm{d}z) \\ &\qquad + \int_0^t\int_0^\infty q_{t-s}(y,z)v(\tau+s,x,z)\,W^\tau(\mathrm{d}s,\mathrm{d}z). \end{aligned}$$

Therefore,

$$\begin{aligned} &J^\tau(t,x,y) + I^\tau(t,x,y) \\ &\quad = \int_0^\infty q_t(y,y')\left(\frac{p^*_\tau(x,y)}{xy} + \int_0^\tau\int_0^\infty q_{\tau-s}(y',z)v(s,x,z)\,W(\mathrm{d}s,\mathrm{d}z)\right)\mathrm{d}y' \\ &\qquad + \int_0^t\int_0^\infty q_{t-s}(y,z)v(\tau+s,x,z)\,W^\tau(\mathrm{d}s,\mathrm{d}z), \end{aligned}$$

as required. □

Now define

$$V(t,x,y) = \begin{cases} v(t,x,y) & \text{if } 0 \le t \le \tau, \\ v^{\tau}(t-\tau,x,y) & \text{if } t > \tau. \end{cases}$$

then $V(t,x,y)$ solves equation (2.3). Let $M > 1$ and $p \ge 2$ then since $\sup_{x,y\in\mathbb{R}_+} \|v(\tau,x,y)\|_p$ is finite, Lemmata 2.4.2 and 2.4.3 applies to show that there is a constant $C := C(M,p,\tau)$ such that for all $x \in \mathbb{R}_+$, y, $y' \in [0,M]$ and t, $t' \in [\tau, M]$

$$\|v^{\tau}(t,x,y) - v^{\tau}(t',x,y')\|_p \le C\big(|t-t'|^{1/4} + |y-y'|^{1/2}\big). \tag{2.34}$$

We now consider the increment in the x variable. In the previous section we have shown that the solution to (2.3) is given by the chaos expansion

$$v(t,x,y) = \frac{p_t^*(x,y)}{xy}\left(1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}_+^k} R_k(\mathbf{s},\mathbf{z};t,x,y)\, W^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{z})\right),$$

where for $0 < s_1 < \cdots < s_k < t$ and $\mathbf{z} = (z_1,\ldots,z_k) \in \mathbb{R}_+^k$

$$R_k(\mathbf{s},\mathbf{z};t,x,y) = \frac{p_{s_1}^*(z_1,x)\prod_{i=2}^k p_{s_i - s_{i-1}}^*(z_i,z_{i-1}) p_{t-s_k}^*(y,z_k)}{p_t^*(y,x)}.$$

Since $p_t^*(x,y) = p_t^*(y,x)$ for all (t,x,y), it is easy to see that the function R_k satisfies

$$R_k(\mathbf{s},\mathbf{z};t,x,y) = R_k(t-\tilde{\mathbf{s}},\tilde{\mathbf{z}};t,y,x), \tag{2.35}$$

where $t - \tilde{\mathbf{s}} := (t-s_k,\ldots,t-s_1)$, $0 < t-s_k < \ldots < t-s_1 < t$ and $\tilde{\mathbf{z}} := (z_k, z_{k-1},\ldots,z_1)$. Using this we have the following

Lemma 2.4.5. *For all $y \in \mathbb{R}_+$, the random fields $(v(t,x,y);(t,x) \in (0,\infty)\times\mathbb{R}_+)$ and $(v(t,y,x);(t,x) \in (0,\infty)\times\mathbb{R}_+)$ are equal in distribution.*

The above lemma follows from Lemma 3.4.4 and the property (2.35) and is proved in exactly the same way as Proposition 3.4.3 below, so we therefore omit it.

Finally, using the above lemma and (2.34), we have for all $M > 1$, $p \ge 2$ there is a constant $C := C(M,p,\tau)$ such that for all (t,x,y), $(t',x',y') \in [2\tau,M]\times[0,M]\times[0,M]$

$$\begin{aligned}
&\|V(t,x,y) - V(t',x',y')\|_p \\
&\qquad \le \|v^{\tau}(t-\tau,x,y) - v^{\tau}(t'-\tau,x,y')\|_p + \|v^{\tau}(t'-\tau,y',x) - v^{\tau}(t'-\tau,y',x')\|_p \\
&\qquad \le C\big(|t-t'|^{1/4} + |x-x'|^{1/2} + |y-y'|^{1/2}\big).
\end{aligned}$$

Since $\tau > 0$ is arbitrary, we can take $2\tau = 1/M$ and thus we have shown that there exists a constant $\tilde{C} := \tilde{C}(M,p)$ such that for all (t,x,y) and $(t',x',y') \in [1/M,M]\times[0,M]\times[0,M]$ the above inequality holds with $\tilde{C}$ in place of C. Finally, an application of Theorem 2.4.1 completes the entire proof of Theorem 2.1.1.

Chapter 3

A Multi-layer Equation

3.1 Introduction

In [OW11] O'Connell and Warren introduced the following: for each $n = 1, 2, \dots$, $t > 0$ and x, $y \in \mathbb{R}$ define

$$Z_n(t, x, y) = p_t(x-y)^n \left(1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s}, \mathbf{y}'; t, x, y)\, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}')\right), \tag{3.1}$$

where $\Delta_k(t) = \{0 < s_1 < s_2 < \cdots < s_k < t\}$, $\mathbf{s} = (s_1, \dots, s_k)$, $\mathbf{y}' = (y_1', \dots, y_k')$ and $R_k(\mathbf{s}, \mathbf{y}'; t, x, y)$ is the k-point correlation function for a collection of n non-intersecting Brownian bridges each of which starts at x at time 0 and ends at y at time t, see Section 3.2.1. $p_t(x-y)$ is the heat kernel $(2\pi t)^{-1/2} e^{-(x-y)^2/2t}$. The integral is a k-fold stochastic integral with respect to space-time white noise, see Appendix A for the definition of such integrals. It was shown in [OW11] by considering local times of non-intersecting Brownian bridges that the infinite sum in the definition is convergent in L^2 with respect to the white noise.

Observe that $u = Z_1$ is the solution to the (multiplicative) stochastic heat equation (SHE) with delta initial data:

$$\begin{cases} \partial_t u(t, x, y) = \left(\frac{1}{2}\Delta_y + \dot{W}(t, y)\right) u(t, x, y), & t \in (0, \infty), y \in \mathbb{R}, \\ u(0, x, y) = \delta(x - y), & x \in \mathbb{R}. \end{cases} \tag{3.2}$$

By a solution to the above we mean a random field u which satisfies almost surely the mild form

$$u(t, x, y) = p_t(x-y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y - y') u(s, x, y')\, W(\mathrm{d}s, \mathrm{d}y'). \tag{3.3}$$

Iterating equation (3.3) multiple times gives the chaos expansion (3.1) for $n = 1$. One can express the solution $u(t,x,y)$ in a more suggestive notation:

$$u(t,x,y) = p_t(x-y)\mathbb{E}^b_{x,y;t}\Big[\mathscr{E}\mathrm{xp}\Big(\int_0^t W(s,b_s)\,\mathrm{d}s\Big)\Big], \tag{3.4}$$

where b is a Brownian bridge that starts at x at time 0 and ends at y at time t and $\mathbb{E}^b_{x,y;t}$ denotes the corresponding expectation. $\mathscr{E}\mathrm{xp}$ is the *Wick exponential* defined by

$$\mathscr{E}\mathrm{xp}(M_t) := \exp\big(M_t - \frac{1}{2}\langle M, M\rangle_t\big),$$

for a martingale M. The Feynman–Kac formula (3.4) is not rigorous as it is unclear how one would define the integral of the white noise along a Brownian path and moreover to exponentiate such an expression. However, Taylor expanding the exponential, then switching the expectation with the infinite sum and evaluating the expectation, one obtains the chaos expansion of u. With this in mind, (3.4) can be thought of as a short hand for the chaos expansion (3.1) in the case $n = 1$. On the other hand, one can obtain a rigorous expression by replacing W in (3.4) with a smoothed version of the space-time white noise. Indeed, Bertini and Cancrini showed in [BC95] that such an expression has a meaningful limit as one takes away the smoothing and that the limit solves the SHE. With this Feynman–Kac interpretation, one can think of the solution to the stochastic heat equation as the partition function (up to a multiplication by the heat kernel) of the continuum directed random polymer [AKQ14a].

Analogously, we write

$$Z_n(t,x,y) = p_t(x-y)^n\mathbb{E}^X_{x,y;t}\Big[\mathscr{E}\mathrm{xp}\Big(\sum_{i=1}^n\int_0^t W(s,X^i_s)\,\mathrm{d}s\Big)\Big], \tag{3.5}$$

where $(X^1_s,\dots,X^n_s, 0 \le s \le t)$ denotes the trajectories of the above mentioned collection of n non-intersecting Brownian bridges and $\mathbb{E}^X_{x,y;t}$ is the corresponding expectation. In the same manner as in the $n = 1$ case, (3.5) should be thought of as the short hand for the chaos expansion (3.1). Therefore, in view of (3.5) one can interpret Z_n as the partition function (up to a factor of the heat kernel) of a natural extension of the continuum directed random polymer involving multiple non-intersecting Brownian paths.

The main result of this and the next chapter is that the continuum partition functions has nice regularity properties.

Theorem 3.1.1. *For all $n \ge 1$, the function $(t,x,y) \mapsto Z_n(t,x,y)$ has a version that is continuous over $(0,\infty)\times\mathbb{R}\times\mathbb{R}$. Moreover,*

$$\mathbb{P}[Z_n(t,x,y) > 0 \textit{ for all } t > 0 \textit{ and } x,y \in \mathbb{R}] = 1.$$

The continuity and strict positivity of $u = Z_1$ was proved by considering its mild form which suggests that to prove Theorem 3.1.1 one could consider the evolution equation satisfied by Z_n. By considering a smooth space-time potential, the authors in [OW11] showed that Z_n should satisfy a certain SPDE, see [OW11, Proposition 3.3 and 3.7], however unfortunately it is not immediately obvious that this SPDE makes sense in the white noise setting. Instead, we shall show that a natural extension of Z_n does satisfy a rigorous evolution equation which can be regarded as a multi-dimensional stochastic heat equation. This allows us to derive the continuity and strict positivity of the extension and from which Theorem 3.1.1 follows as a corollary.

Denote by W_n the Weyl chamber $\{\mathbf{x} \in \mathbb{R}^n : x_1 \geq x_2 \cdots \geq x_n\}$, then for $n = 1, 2, \ldots$, $t > 0$ and $\mathbf{x}$, $\mathbf{y} \in W_n$ define

$$K_n(t, \mathbf{x}, \mathbf{y}) = p_n^*(t, \mathbf{x}, \mathbf{y}) \left(1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s}, \mathbf{y}'; t, \mathbf{x}, \mathbf{y}) \, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}') \right), \tag{3.6}$$

where R_k is the k-point correlation function of a collection of n non-intersection Brownian bridges which starts at $\mathbf{x}$ at time 0 and ends at $\mathbf{y}$ at time t. $p_n^*(t, \mathbf{x}, \mathbf{y}) = \det[p_t(x_i - y_j)]_{i,j=1}^n$ is by the Karlin–McGregor formula [KM59] the transition density of Brownian motion killed at the boundary of W_n. It was proved in [OW11, Proposition 3.2] that K_n also satisfies a Karlin–McGregor type formula:

$$K_n(t, \mathbf{x}, \mathbf{y}) = \det[u(t, x_i, y_j)]_{i,j=1}^n, \tag{3.7}$$

where each term in the determinant are solutions to (3.2) each driven by the same white noise. Now, define for $t > 0$, $\mathbf{x}$, $\mathbf{y} \in W_n^\circ$

$$M_n(t, \mathbf{x}, \mathbf{y}) = \frac{K_n(t, \mathbf{x}, \mathbf{y})}{\Delta(\mathbf{x})\Delta(\mathbf{y})}, \tag{3.8}$$

where $\Delta(\mathbf{x}) = \prod_{1 \leq i < j \leq n}(x_i - x_j)$ is the Vandermonde determinant. It follows from (3.6) that M_n has chaos expansion

$$M_n(t, \mathbf{x}, \mathbf{y}) = \frac{p_n^*(t, \mathbf{x}, \mathbf{y})}{\Delta(x)\Delta(y)} \left(1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s}, \mathbf{y}'; t, \mathbf{x}, \mathbf{y}) \, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}') \right). \tag{3.9}$$

By (3.7) and the continuity of the solution to the stochastic heat equation, it is easy to see that $K_n(t, \mathbf{x}, \mathbf{y})$ is almost surely continuous on $(0, t) \times W_n \times W_n$ and is zero on the boundary of $W_n \times W_n$. It follows that $M_n(t, \mathbf{x}, \mathbf{y})$ is continuous in the interior $W_n^\circ \times W_n^\circ$. By [BBO09, Lemma 5.11], $p_n^*(t, \mathbf{x}, \mathbf{y})/\Delta(\mathbf{x})\Delta(\mathbf{y})$ is a smooth function of $(\mathbf{x}, \mathbf{y})$ over $\mathbb{R}^n \times \mathbb{R}^n$ and since the k-point correlation function R_k extends continuously to the boundary of the Weyl chamber, see Section 3.2.1, we see from its chaos expansion (3.9) that $M_n(t, \mathbf{x}, \mathbf{y})$ is defined for $\mathbf{x}$, $\mathbf{y} \in \partial W_n$. This also suggests that $M_n(t, \mathbf{x}, \mathbf{y})$ is a continuous function on $W_n \times W_n$. Furthermore, from (3.7) we see that M_n being a ratio of determinants is a permutation symmetric function of its spatial variables, that is for any permutations π,

σ of $\{1,\ldots,n\}$, $M_n(t,\pi\mathbf{x},\sigma\mathbf{y}) = M_n(t,\mathbf{x},\mathbf{y})$. Hence, we can extend M_n by symmetry to a function on $\mathbb{R}^n \times \mathbb{R}^n$ and we will show here and in the next chapter that there exists a version of M_n that is almost surely strictly positive and continuous on the whole of $\mathbb{R}^n \times \mathbb{R}^n$ and for all $t > 0$. Moreover, when all the $\mathbf{x}$ coordinates are equal and likewise for $\mathbf{y}$, M_n agrees up to a multiplicative constant with Z_n, that is

$$M_n(t, a\mathbf{1}, b\mathbf{1}) = c_{n,t} Z_n(t,a,b), \tag{3.10}$$

where $c_{n,t} := \left(\prod_{i=1}^{n-1} i!\right)^{-1} t^{-n(n-1)/2}$ and $\mathbf{1} = (1,\ldots,1)$. Equation (3.10) was shown to hold in [OW11] but there the continuity of M_n on the boundary of W_n was only established in an L^2 sense; here we extend it to almost sure continuity. Note that (3.7) suggests that $K_n(t,\mathbf{x},\mathbf{y})$ and $M_n(t,\mathbf{x},\mathbf{y})$ can be regarded as the stochastic analogue of $p_n^*(t,\mathbf{x},\mathbf{y})$ and $p_n^*(t,\mathbf{x},\mathbf{y})/\Delta(\mathbf{x})\Delta(\mathbf{y})$ respectively where the latter has limit at the boundary equal to $c_{n,t}p_t(a-b)^n$.

In Section 3.3, we will show that for all $(t,\mathbf{x},\mathbf{y}) \in (0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$, $M_n(t,\mathbf{x},\mathbf{y})$ satisfies almost surely the mild equation

$$\begin{aligned} M_n(t,\mathbf{x},\mathbf{y}) &= \frac{p_n^*(t,\mathbf{x},\mathbf{y})}{\Delta(\mathbf{x})\Delta(\mathbf{y})} + A_n \int_0^t \int_{\mathbb{R}^n} Q_{t-s}(\mathbf{y},\mathbf{y}') M_n(s,\mathbf{x},\mathbf{y}')\, \mathrm{d}\mathbf{y}'_*\, W(\mathrm{d}s, \mathrm{d}y'_1) \\ &=: J_n(t,\mathbf{x},\mathbf{y}) + I_n(t,\mathbf{x},\mathbf{y}), \end{aligned} \tag{3.11}$$

where $A_n = 1/(n-1)!$ is a combinatorial constant, $\mathrm{d}\mathbf{y}'_* = \mathrm{d}y_2 \ldots \mathrm{d}y_n$ and

$$Q_t(\mathbf{x},\mathbf{y}) = \frac{\Delta(\mathbf{y})}{\Delta(\mathbf{x})} p_n^*(t,\mathbf{x},\mathbf{y}),$$

is the transition density of Dyson's Brownian motion starting from $\mathbf{x} \in W_n$ and ending at $\mathbf{y} \in W_n$. It satisfies

$$Q_t(a\mathbf{1},\mathbf{y}) = c_{n,t}\Delta(\mathbf{y})^2 \prod_{i=1}^n p_t(y_i - a). \tag{3.12}$$

We can extend Q_t by symmetry to a function on $\mathbb{R}^n \times \mathbb{R}^n$ and so the integral over $\mathbb{R}^n$ in the mild equation (3.11) is defined.

Consider also the following integral equation for $(t,\mathbf{y}) \in (0,\infty)\times\mathbb{R}^n$,

$$\begin{aligned} M_n(t,\mathbf{y}) = \frac{1}{n!} &\int_{\mathbb{R}^n} g(\mathbf{y}') Q_t(\mathbf{y},\mathbf{y}')\, \mathrm{d}\mathbf{y}' \\ &+ A_n \int_0^t \int_{\mathbb{R}^n} Q_{t-s}(\mathbf{y},\mathbf{y}') M_n(s,\mathbf{y}')\, \mathrm{d}\mathbf{y}'_*\, W(\mathrm{d}s,\mathrm{d}y'_1) \\ =&: J_n(t,\mathbf{y}) + I_n(t,\mathbf{y}), \end{aligned} \tag{3.13}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is permutation symmetric and may be random but independent of the white noise. The function g is the initial condition for equation (3.13) in the sense that

$$\lim_{t\to 0}\frac{1}{n!}\int_{\mathbb{R}^n} g(\mathbf{y}')Q_t(\mathbf{y},\mathbf{y}')\,\mathrm{d}\mathbf{y}' = \lim_{t\to 0}\int_{W_n} g(\mathbf{y}')Q_t(\mathbf{y},\mathbf{y}')\,\mathrm{d}\mathbf{y}' = g(\mathbf{y}).$$

On the other hand, we say that $M_n(t,\mathbf{x},\mathbf{y})$ is the solution started from a delta initial data at $\mathbf{x}$ even though strictly speaking it is the ratio of $K_n(t,\mathbf{x},\mathbf{y})$, which can be shown to satisfy an integral equation similar to (3.13) with delta initial condition, and the product of Vandermonde determinants $\Delta(\mathbf{x})\Delta(\mathbf{y})$. To emphasise the initial data we sometimes write $M_n^g(t,\mathbf{y})$ instead of $M_n(t,\mathbf{y})$. Equations (3.13) and (3.11) are the multi-dimensional counterpart to equations (2.1) and (2.3) of the previous chapter respectively.

We now state the main results regarding the solutions of (3.11) and (3.13) from which the continuity part of Theorem 3.1.1 follows as a corollary by (3.10). Let's first set up the probability space in which we work. Let $\mathscr{B}_b(\mathbb{R})$ be the collection of Borel measurable subsets of $\mathbb{R}$ with finite Lebesgue measure and let $W = \big(W_t(A), t\geq 0, A\in\mathscr{B}_b(\mathbb{R})\big)$ be space-time white noise on a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$ endowed with a right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$ such that W is $\mathscr{F}_t$-adapted and $W_t(A)-W_s(A)$ is independent of $\mathscr{F}_s$ for all $A\in\mathscr{B}_b(\mathbb{R})$. From now on we fix this filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$. We use $\mathbb{E}$ to denote the expectation with repect to $\mathbb{P}$ and for $p\geq 1$, $\|\cdot\|_p = (\mathbb{E}[|\cdot|^p])^{1/p}$ denotes the $L^p(\Omega)$ norm. Throughout, $c_p \leq 2\sqrt{p}$ is the constant appearing in the Burkholder–Davis–Gundy inequality.

Theorem 3.1.2. *(a) Suppose that g is $\mathscr{F}_0$-measurable and symmetric and satisfies for all $p\geq 2$, $\sup_{\mathbf{y}\in\mathbb{R}^n}\|g(\mathbf{y})\|_p \leq K_{p,g} < \infty$, then there exists a solution $\big(M_n(t,\mathbf{y}),(t,\mathbf{y})\in[0,\infty)\times\mathbb{R}^n\big)$ to the integral equation (3.13) that is unique (in the sense of versions) in the class of all random fields $\big(v(t,\mathbf{y}),(t,\mathbf{y})\in[0,\infty)\times\mathbb{R}^n\big)$ that satisfy $\sup_{(t,\mathbf{y})\in[0,T]\times\mathbb{R}^n}\|v(t,\mathbf{y})\|_p<\infty$ for all $T>0$. The solution satisfies for all $p\geq 2$*

$$\|M_n(t,\mathbf{y})\|_p^2 \leq 2K_{p,g}^2 e^{A^2c_p^4 t}\big(1+\mathrm{erf}(Ac_p^2t^{1/2})\big), \tag{3.14}$$

for a constant $A>0$ depending on n.

Moreover, M_n has a version such that $(t,\mathbf{y})\mapsto M_n(t,\mathbf{y})$ is locally Hölder continuous on $(0,\infty)\times\mathbb{R}^n$ with indices $\alpha<1/2$ in space and $\alpha<1/4$ in time.

(b) There exists a unique solution $\big(M_n(t,\mathbf{x},\mathbf{y})\in(0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\big)$ given by the chaos expansion (3.9) to the integral equation (3.11) such that for all $p\geq 2$, $t>0$ and $\mathbf{x}$, $\mathbf{y}\in W_n$

$$\|M_n(t,\mathbf{x},\mathbf{y})\|_p^2 \leq 2\left(\frac{p_n^*(t,\mathbf{x},\mathbf{y})}{\Delta(\mathbf{x})\Delta(\mathbf{y})}\right)^2 \mathbb{E}_{x,y;t}^{X,Y}\Big[\exp\Big(2c_p^2\sum_{i,j=1}^n L_t(X^i-Y^j)\Big)\Big], \tag{3.15}$$

where $L_t(X^i-Y^j)$ is the local time at 0 of the difference X^i-Y^j with X^i being the ith component of the collection of n non-intersecting Brownian bridges X started at $\mathbf{x}$

and ending at $\mathbf{y}$ *at time* t *and* Y *is an independent copy of the bridge.* $\mathbb{E}^{X,Y}_{x,y;t}$ *denotes the corresponding expectation of the joint law of the bridges.*

Moreover, M_n *has a version such that* $(t, \mathbf{x}, \mathbf{y}) \mapsto M_n(t, \mathbf{x}, \mathbf{y})$ *is locally Hölder continuous on* $(0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$ *with indices* $\alpha < 1/2$ *in space and* $\alpha < 1/4$ *in time.*

We follow essentially the same strategy as in the proof of Theorem 2.1.1 in Chapter 2 to prove the above theorem and this means we require certain bounds and continuity estimates for the kernel Q_t (see Theorem 3.2.5 below). The continuity result in part (a) is simpler to obtain as all the pth moments of the solution are bounded uniformly in space and time. The solution in part (b) have pth moments that blow up at small times but however they are bounded uniformly in space for each fixed positive time and so in the same way as for the proof of continuity in Theorem 2.1.1(b) we can solve equation (3.11) up to a small arbitrary time and restart the equation and apply the continuity estimates of part (a). In part (b) we will also establish the continuity in the $\mathbf{x}$ variable by exploiting a symmetry property in $\mathbf{x}$ and $\mathbf{y}$ of $M_n(t, \mathbf{x}, \mathbf{y})$ which allows us to apply the continuity estimate in the $\mathbf{y}$ variable to get the corresponding estimate for the $\mathbf{x}$ variable.

The outline of the chapter is as follows. In Section 3.2.1 we discuss non-intersecting Brownian motions and in Sections 3.2.2 and 3.2.3 we prove a version of Lemma 2.2.11 and Proposition 2.2.10 adapted to the present multi-dimensional setting. Then in Section 3.2.4 we prove certain continuity estimates for Q_t which are essential in the proof of continuity of M_n. The existence, uniqueness and moment estimates part of Theorem 3.1.2 will be proved in Section 3.3 and finally the continuity result will be proved in Section 3.4.

3.2 Preliminaries

3.2.1 Non-intersecting Brownian Motions

Dyson Brownian motion introduced in [Dys62] can be realised as the eigenvalues of Hermitian Brownian motion, an $n \times n$ Hermitian matrix whose entries are (up to the Hermitian condition) independent standard complex Brownian motions. The eigenvalues of such a matrix is a Markov process with state space W_n with transition density $Q_t(\mathbf{x}, \mathbf{y})$. It also arises as the Doob h-transform of Brownian motion killed at the boundary ∂W_n with $h(\mathbf{x}) = \Delta(\mathbf{x})$ (see for example [Gra99] and [KT07]).

One can construct bridges of Dyson Brownian motion, which we will call Dyson Brownian bridge or non-intersecting Brownian bridges, using the framework of [FPY93]. For $\mathbf{x}, \mathbf{y} \in W_n$, a collection of non-intersecting Brownian bridges $X_s = (X^1_s, \dots, X^n_s)$ starting at $\mathbf{x}$ at time 0 and ending at $\mathbf{y}$ in time t is a process whose law is absolutely continuous with respect to that of Dyson Brownian motion started at $\mathbf{x}$ with Radon–Nikodym derivative equal to

$$\frac{Q_{t-s}(X_s, \mathbf{y})}{Q_t(\mathbf{x}, \mathbf{y})}.$$

In particular, for $0 < s_1 < \ldots < s_k < t$, the law of $(X_{s_1}, \ldots, X_{s_k})$ is given by the density

$$\frac{Q_{s_1}(\mathbf{x}, \mathbf{y}^1) \prod_{i=2}^{k} Q_{s_i - s_{i-1}}(\mathbf{y}^{i-1}, \mathbf{y}^i) Q_{t-s_k}(\mathbf{y}^k, \mathbf{y})}{Q_t(\mathbf{x}, \mathbf{y})}$$

The above is well defined at the boundary of the Weyl chamber by (3.12); in particular, taking limits as $\mathbf{x} \to a\mathbf{1}$, $\mathbf{y} \to b\mathbf{1}$ where $\mathbf{1} = (1, \ldots, 1)$ one obtains

$$c_n \frac{\Delta(\mathbf{y}^1)\Delta(\mathbf{y}^k) \prod_{j=1}^{n} p_{s_1}(a - y_j^1) \prod_{i=2}^{k} p_n^*(s_i - s_{i-1}, \mathbf{y}^{i-1}, \mathbf{y}^i) \prod_{j=1}^{n} p_{t-s_k}(b - y_j^k)}{s_1^{n(n-1)/2}(t - s_k)^{n(n-1)/2} t^{-n(n-1)/2} p_t(a-b)^n},$$

where $c_n^{-1} = \prod_{i=1}^{n-1} i!$. The k-point correlation function R_k appearing in (3.6) is defined as the sum over $i_1, \ldots, i_k$ for $1 \leq i_r \leq n$, $1 \leq r \leq k$ of the densities of the process $(X_{s_1}^{i_1}, \ldots, X_{s_k}^{i_k})$:

$$R_k(\mathbf{s}, \sum_{i_1, \ldots, i_k} \int_{(W_{n-1})^k} \frac{p_n^*(s_1, \mathbf{x}, \mathbf{y}^1) \prod_{i=2}^{k} p_n^*(s_i - s_{i-1}, \mathbf{y}^{i-1}, \mathbf{y}^i) p_n^*(t - s_k, \mathbf{y}^k, \mathbf{y})}{p_n^*(t, \mathbf{x}, \mathbf{y})} \prod_{l=1}^{k} \prod_{j \neq i_l}^{n} \mathrm{d}y_j^l$$

Notice that the integrand above is symmetric in the permutation of its arguments $(y_1^l, \ldots, y_n^l)$ for all $1 \leq l \leq k$ and so we can rewrite each integral over W_{n-1} as integrals over $\mathbb{R}^{n-1}$ multiplied by a factor of $1/n!$. Moreover, by symmetry each term in the sum over $i_1, \ldots, i_k$ gives the same contribution. There are in total n^k of such k-tuples and hence we can rewrite the correlation function $R_k\big((s_1, y_1^1), \ldots, (s_k, y_1^k); t, \mathbf{x}, \mathbf{y}\big) := R_k(\mathbf{s}, \mathbf{y}_1; t, \mathbf{x}, \mathbf{y})$, $\mathbf{y}_1 = (y_1^1, \ldots, y_1^k)$ as

$$A_n^k \int_{(\mathbb{R}^{n-1})^k} \frac{p_n^*(s_1, \mathbf{x}, \mathbf{y}^1) \prod_{i=2}^{k} p_n^*(s_i - s_{i-1}, \mathbf{y}^{i-1}, \mathbf{y}^i) p_n^*(t - s_k, \mathbf{y}^k, \mathbf{y})}{p_n^*(t, \mathbf{x}, \mathbf{y})} \prod_{i=1}^{k} \prod_{j=2}^{n} \mathrm{d}y_j^i, \quad (3.16)$$

where $A_n := 1/(n-1)!$. For each k we have chosen to leave the first coordinate of $\mathbf{y}^k$ and integrated out the rest but this choice is arbitrary by symmetry. Note that this is also the reason for the form of the stochastic integral term in (3.11).

In the sequel we will need to bound integrals of the square of the k-point correlation function R_k. Correlation functions of densities given by a product of determinants have been studied extensively in the context of determinantal point processes, see for example [Joh06] and [Bor11]. They can be expressed as a determinant of a matrix whose entries are given by some kernel function. However for general start and end points $\mathbf{x}$ and $\mathbf{y}$ this kernel function is difficult to compute, but since all we need is the integral of the square of R_k it is not necessary to compute R_k explicitly and so we will not pursue this. Instead, the next two results proved in [OW11], which express the integral of R_k^2 in terms of intersection local times of Brownian bridges, will be used. Let $X = (X^1, \ldots, X^n)$ and $Y = (Y^1, \ldots, Y^n)$ be two independent copies of a collection of n non-intersecting Brownian bridges which start at $\mathbf{x}$ at time 0 and end at $\mathbf{y}$ at time t and let $\mathbb{E}_{\mathbf{x},\mathbf{y};t}^{X,Y}$ denote the corresponding expectation of the joint law of the bridges. Let $L_t(X^i - Y^j)$ be the local time at 0 of the difference

$X^i - Y^j$. Then we have

Lemma 3.2.1 (Lemma 4.1 of [OW11])**.** *Fix $n \geq 1$. For all integers $k \geq 1$ and all $t > 0$, $\mathbf{x}$, $\mathbf{y} \in W_n$ the following holds*

$$\int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s}, \mathbf{y}'; t, \mathbf{x}, \mathbf{y})^2 \, \mathrm{d}\mathbf{y}' \mathrm{d}\mathbf{s} = \frac{1}{k!} \mathbb{E}^{X,Y}_{\mathbf{x},\mathbf{y};t} \Big[\Big(\sum_{i,j=1}^{n} L_t(X^i - Y^j) \Big)^k \Big].$$

The following is used to bound the above moments of local times.

Lemma 3.2.2 (Proposition 4.2 of [OW11])**.** *For all $a \geq 1$ and $0 < t \leq T$, there exists a constant $C := C(a, n, T) > 0$ such that*

$$\sup_{\mathbf{x},\mathbf{y} \in W_n} \left(\frac{p_n^*(t, \mathbf{x}, \mathbf{y})}{\Delta(\mathbf{x})\Delta(\mathbf{y})} \right)^2 \mathbb{E}^{X,Y}_{\mathbf{x},\mathbf{y};t} \Big[\exp \Big(a \sum_{i,j=1}^{n} L_t(X^i - Y^j) \Big) \Big] \leq C t^{-n^2}.$$

The above two lemmata show that for each $t > 0$, $\|Z_n(t, x, y)\|_2 < \infty$ uniformly in x and y and thus the chaos series (3.1) is convergent in $L^2(\Omega)$. The same is also true for the chaos series (3.6).

3.2.2 L^p Bounds on Stochastic Integrals

The following estimate is a useful bound on the $L^p(\Omega)$ norm of stochastic integrals; it can be considered as a version of Lemma 2.2.11 adapted to the present setting. Recall that for brevity we denote $\mathrm{d}\mathbf{y}'_* = \mathrm{d}y'_2 \ldots \mathrm{d}y'_n$ and $c_p \leq 2\sqrt{p}$ is the constant appearing in the Burkhoider–Davis–Gundy inequality.

Lemma 3.2.3. *Define a random field $\big(f(t, \mathbf{y}); (t, \mathbf{y}) \in (0, \infty) \times \mathbb{R}^n\big)$ by*

$$f(t, \mathbf{y}) = \int_0^t \int_{\mathbb{R}^n} \Gamma_{t-s}(\mathbf{y}, \mathbf{y}') w(s, \mathbf{y}') \, \mathrm{d}\mathbf{y}'_* \, W(\mathrm{d}s, \mathrm{d}y'_1),$$

for a suitable random field w and $\Gamma_t(\mathbf{y}, \mathbf{y}')$ is a non-random measurable function on $(0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$ such that $\int_{\mathbb{R}^{n-1}} \Gamma_{t-s}(\mathbf{y}, \mathbf{y}') w(s, \mathbf{y}') \, \mathrm{d}\mathbf{y}'_*$ is integrable in the sense of Walsh for all $(t, \mathbf{y}) \in (0, \infty) \times \mathbb{R}^n$. Then for all integers $p \geq 2$, $t \geq 0$ and $\mathbf{y} \in \mathbb{R}^n$

$$\|f(t, \mathbf{y})\|_p^2 \leq c_p^2 \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} \Gamma_{t-s}(\mathbf{y}, \mathbf{y}') \|w(s, \mathbf{y}')\|_p \, \mathrm{d}\mathbf{y}'_* \right)^2 \mathrm{d}y'_1 \mathrm{d}s.$$

Proof. Fix t and $\mathbf{y}$, then by the Burkholder–Davis–Gundy inequality applied to the martingale $\big(\int_0^r \int_{\mathbb{R}^n} \Gamma_{t-s}(\mathbf{y}, \mathbf{y}') w(s, \mathbf{y}') \, \mathrm{d}\mathbf{y}'_* \, W(\mathrm{d}s, \mathrm{d}y'_1),\ r \in [0, t] \big)$, we have

$$\|f(t, \mathbf{y})\|_p^2 \leq c_p^2 \left\| \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} \Gamma_{t-s}(\mathbf{y}, \mathbf{y}') w(s, \mathbf{y}') \, \mathrm{d}\mathbf{y}'_* \right)^2 \mathrm{d}y'_1 \mathrm{d}s \right\|_{p/2}.$$

Applying Minkowski's integral inequality twice, we obtain

$$\begin{aligned}\|f(t,\mathbf{y})\|_p^2 &\le c_p^2 \int_0^t \int_{\mathbb{R}} \left\| \int_{\mathbb{R}^{n-1}} \Gamma_{t-s}(\mathbf{y},\mathbf{y}')w(s,\mathbf{y}')\,\mathrm{d}\mathbf{y}'_* \right\|_p^2 \mathrm{d}y_1'\mathrm{d}s \\ &\le c_p^2 \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} \Gamma_{t-s}(\mathbf{y},\mathbf{y}')\|w(s,\mathbf{y}')\|_p\,\mathrm{d}\mathbf{y}'_* \right)^2 \mathrm{d}y_1'\mathrm{d}s,\end{aligned}$$

as required. □

3.2.3 Predictability of Random Fields

Recall that the Walsh integral is defined for random fields in $\mathscr{P}_2$, see Appendix A. In the sequel we will need to integrate functions of the form: for some random field M, let $f(s,y_1') = \int_{\mathbb{R}^{n-1}} Q_{t-s}(\mathbf{y},\mathbf{y}')M(s,\mathbf{y}')\,\mathrm{d}\mathbf{y}'_*$. (Note that we have suppressed the dependency of f on t and $\mathbf{y}$ to keep the notation simple). The following proposition provides convenient conditions to verify the integrability of such a random field.

Proposition 3.2.4. *Let $t > 0$ and $\mathbf{y} \in \mathbb{R}^n$. Suppose the random field $\big(M(s,\mathbf{y}'), (s,\mathbf{y}') \in (0,t) \times \mathbb{R}^n\big)$ satisfies*

(i) M is adapted i.e., for all $(s,\mathbf{y}') \in (0,t)\times\mathbb{R}^n$, $M(s,\mathbf{y}')$ is $\mathscr{F}_s$-measurable;

(ii) $(s,\mathbf{y}') \mapsto M(s,\mathbf{y}')$ is $L^2(\Omega)$-continuous on $(0,t)\times\mathbb{R}^n$;

(iii) $\sup_{(s,\mathbf{y}')\in(0,t)\times\mathbb{R}^n} \|M(s,\mathbf{y}')\|_2 < \infty$;

Then $\big(f(s,z), (s,z) \in (0,t)\times\mathbb{R}\big)$ defined by $f(s,y_1') = \int_{\mathbb{R}^{n-1}} Q_{t-s}(\mathbf{y},\mathbf{y}')M(s,\mathbf{y}')\,\mathrm{d}\mathbf{y}'_*$ is in $\mathscr{P}_2$ and

$$\int_0^t \int_{\mathbb{R}} f(s,y_1')\, W(\mathrm{d}s,\mathrm{d}y_1'),$$

is a well-defined Walsh integral.

Proof. We will show that f satisfies the three assumptions of Proposition 2.2.10 in Chapter 2. Since $Q_{t-s}(\mathbf{y},\mathbf{y}')$ is continuous and deterministic, $Q_{t-s}(\mathbf{y},\mathbf{y}')M(s,\mathbf{y}')$ is adapted by (i) and so the integral $\int_{\mathbb{R}^{n-1}} Q_{t-s}(\mathbf{y},\mathbf{y}')M(s,\mathbf{y}')\,\mathrm{d}\mathbf{y}'_*$ is also adapted. Assumption (iii) of Proposition 2.2.10 follows from (iii) above since by Lemma 3.2.3 and Lemma 3.2.11 below, we have for some constant C

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}} \|f(s,y_1')\|_2^2\,\mathrm{d}y_1'\mathrm{d}s \\ &\qquad \le \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} Q_{t-s}(\mathbf{y},\mathbf{y}')\|M(s,\mathbf{y}')\|_2\,\mathrm{d}\mathbf{y}'_* \right)^2 \mathrm{d}y_1'\mathrm{d}s \\ &\qquad \le \sup_{(s,\mathbf{y}')\in(0,t)\times\mathbb{R}^n} \|M(s,\mathbf{y}')\|_2^2 \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} Q_{t-s}(\mathbf{y},\mathbf{y}')\,\mathrm{d}\mathbf{y}'_* \right)^2 \mathrm{d}y_1'\mathrm{d}s \\ &\qquad \le Ct^{1/2} \sup_{(s,\mathbf{y}')\in(0,t)\times\mathbb{R}^n} \|M(s,\mathbf{y}')\|_2^2.\end{aligned}$$

It remains to show the $L^2(\Omega)$-continuity of f. We wish to show that for each $(s,y) \in (0,t)\times\mathbb{R}$, $\lim_{(u,z)\to(s,y)} \|f(u,z) - f(s,y)\|_2 = 0$. Let $h > 0$ and suppose $z_1 \in [y_1' - h, y_1' + h]$ and $u \in [s/2, (t+s)/2]$. Then by the Harish-Chandra formula (3.17) and equation (3.18) below, we have

$$\begin{aligned} Q_{t-u}(\mathbf{y},\mathbf{z}) &\leq c_n (t-u)^{-n^2/2} \Delta(\mathbf{z})^2 \prod_{i=1}^n e^{-(y_i - z_i)^2/2(t-u)} \\ &\leq \frac{2^{n^2/2} c_n}{(t-s)^{n^2/2}} \prod_{2\leq i<j\leq n} (z_i - z_j)^2 \prod_{i=2}^n (|y_1' + h| + |z_i|)^2 e^{-\frac{(y_i - z_i)^2}{2(t-s/2)}} e^{-\frac{y_1^2 - 2|y_1' + h| y_1}{2(t-s/2)}}. \end{aligned}$$

The last line is integrable with respect to $\mathrm{d}\mathbf{z}_* = \mathrm{d}z_2 \ldots \mathrm{d}z_n$ and so by the dominated convergence theorem, the continuity of Q_t and assumption (ii), the right hand side of

$$\begin{aligned} \|f(u,z_1) - &f(s,y_1')\|_2 \\ &\leq \sup_{(s,\mathbf{y})} \|M(s,\mathbf{y})\|_2 \int_{\mathbb{R}^{n-1}} \left| Q_{t-u}\big(\mathbf{y},(z_1,\mathbf{z}_*)\big) - Q_{t-s}\big(\mathbf{y},(y_1',\mathbf{z}_*)\big) \right| \mathrm{d}\mathbf{z}_* \\ &\quad + \int_{\mathbb{R}^{n-1}} Q_{t-s}\big(\mathbf{y},(y_1',\mathbf{z}_*)\big) \|M\big(u,(z_1,\mathbf{z}_*)\big) - M\big(s,(y_1',\mathbf{z}_*)\big)\|_2 \,\mathrm{d}\mathbf{z}_* \end{aligned}$$

converges to zero as $(u,z_1) \to (s,y_1')$. Finally, an application of Proposition 2.2.10 completes the proof. $\square$

3.2.4 Estimates on Q_t

From now on we drop the bold typeface for vectors in $\mathbb{R}^n$ or W_n since we will only be working with functions of multi-dimensional spatial variables so there is no longer any risk of confusion.

Before proving Theorem 3.1.2 we need estimates on various quantities involving the kernel Q_t. The following known as the Harish-Chandra/Itzykson–Zuber formula [IZ80] provides a useful alternate expression for Q_t:

$$\frac{\det[e^{-(x_i - y_j)/2t}]}{\Delta(x)\Delta(y)} = c_n t^{-n(n-1)/2} \int_{\mathcal{U}(n)} \exp\Big(-\frac{1}{2t}\mathrm{Tr}(Y - UXU^\dagger)^2\Big)\,\mathrm{d}U, \tag{3.17}$$

for Hermitian matrices X and Y with eigenvalues $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. $c_n = \big(\prod_{i=1}^{n-1} i!\big)^{-1}$ and the integral is with respect to the normalised Haar measure on the unitary group $\mathcal{U}(n)$. Furthermore, the integrand above is bounded uniformly in U as the following bound from [MRTZ06, Lemma 1] shows

$$\sup_{U\in\mathcal{U}(n)} \exp\Big(-\frac{1}{2t}\mathrm{Tr}(Y - UXU^\dagger)^2\Big) \leq \prod_{i=1}^n e^{-(y_i - x_i)^2/2t}. \tag{3.18}$$

As mentioned in the introduction, $Q_t(x,y)$ is well defined on the boundary of the Weyl chamber and since it is a product and ratio of determinants, it is permutation symmetric and so we can extend Q_t to a function on $\mathbb{R}^n \times \mathbb{R}^n$ by symmetry. Denote $K_t(x, y_1) := \int_{\mathbb{R}^{n-1}} Q_t(x, y) \prod_{i=2}^n \mathrm{d}y_i$ and $K := K_1$. The following result strongly indicates the continuity of M_n; in fact it is a key estimate in its proof in Section 3.4.

Theorem 3.2.5. *(a) There is a constant $C_1 > 0$ depending only on n such that for all $t > 0$ and $x, z \in \mathbb{R}^n$ we have*

$$\int_0^t \int_{\mathbb{R}} \big(K_s(x, y) - K_s(z, y)\big)^2 \,\mathrm{d}y\mathrm{d}s \leq C_1|x - z|,$$

(b) there are constants C_2, $C_3 > 0$ depending only on n such that for all t, u with $0 < u \leq t < \infty$ and $x \in \mathbb{R}^n$, we have

$$\int_0^u \int_{\mathbb{R}} \big(K_{t-u+s}(x, y) - K_s(x, y)\big)^2 \,\mathrm{d}y\mathrm{d}s \leq C_2|t - u|^{1/2},$$

and

$$\int_u^t \int_{\mathbb{R}} K_s(x, y)^2 \,\mathrm{d}y\mathrm{d}s \leq C_3|t - u|^{1/2}.$$

The theorem is a consequence of the series of results below. First observe that Q_t has the following scaling property:

$$Q_t(x, y) = t^{-n/2} \frac{\Delta(y/\sqrt{t})}{\Delta(x/\sqrt{t})} \det\Big[\frac{1}{\sqrt{2\pi}} e^{-(x/\sqrt{t}-y/\sqrt{t})^2/2}\Big] = t^{-n/2} Q_1(x/\sqrt{t}, y/\sqrt{t}). \tag{3.19}$$

The left hand side of the inequality in Theorem 3.2.5(a) is bounded above by

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^{n-1}} Q_s(x, y) - Q_s(z, y) \prod_{i=2}^n \mathrm{d}y_i \Big)^2 \mathrm{d}y_1\mathrm{d}s \\ &\quad = \int_0^\infty \frac{1}{\sqrt{s}} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^{n-1}} Q_1(x/\sqrt{s}, y') - Q_1(z/\sqrt{s}, y') \prod_{i=2}^n \mathrm{d}y'_i \Big)^2 \mathrm{d}y'_1\mathrm{d}s, \end{aligned} \tag{3.20}$$

where we have changed the integration region to $[0, \infty)$ in the time integral which results in an upper bound due to the positivity of the integrand. The equality follows from the scaling property (3.19) and a change of variables. Theorem 3.2.5(a) now follows from (3.20) and Lemma 2.2.5 in Chapter 2. Thus, in the same way as in the proof of Proposition 2.2.4 we need to show that $K(x, y)$ satisfies the hypothesis of Lemma 2.2.5. Using the inequality $(a + b)^2 \leq 2(a^2 + b^2)$, the left hand side of (2.14) with K in place of R, can be bounded by

$$2\Big(\int_{\mathbb{R}} K(x, y)^2 \,\mathrm{d}y + \int_{\mathbb{R}} K(z, y)^2 \,\mathrm{d}y \Big) \leq 4 \sup_{x \in \mathbb{R}^n} \|K(x, \cdot)\|^2_{L^2(\mathrm{d}y)}.$$

On the other hand, let $r(\rho) : [0, 1] \to \mathbb{R}^n$, $r(\rho) = (1 - \rho)x + \rho z$ be a parameterisation of the straight line from x to z, then

$$K(x,y) - K(z,y) = \int_0^1 \nabla K(r(\rho), y) \cdot r'(\rho) \, \mathrm{d}\rho,$$

where the gradient is with respect to the first variable of $K(\cdot,\cdot)$ and $u \cdot v$ denotes the usual inner product of two vectors in $\mathbb{R}^n$. Then by Minkowski's integral inequality and Cauchy–Schwarz inequality we have

$$\begin{aligned}
\left(\int_{\mathbb{R}} \big(K(x,y) - K(z,y)\big)^2 \, \mathrm{d}y \right)^{1/2} &\leq \int_0^1 \|\nabla K(r(\rho), \cdot) \cdot r'(\rho)\|_{L^2(\mathrm{d}y)} \, \mathrm{d}\rho \\
&\leq \int_0^1 \big\| |\nabla K(r(\rho), \cdot)| \big\|_{L^2(\mathrm{d}y)} \, |r'(\rho)| \, \mathrm{d}\rho \\
&\leq \sup_{\rho \in [0,1]} \big\| |\nabla K(r(\rho), \cdot)| \big\|_{L^2(\mathrm{d}y)} \, |x - z|.
\end{aligned}$$

Therefore, in order to verify the hypothesis of Lemma 2.2.5 we need to show that

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}} K(x,y)^2 \, \mathrm{d}y < \infty, \tag{3.21}$$

and

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}} |\nabla K(x,y)|^2 \, \mathrm{d}y < \infty. \tag{3.22}$$

We first concentrate on (3.22). It suffices to show that

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}} \frac{\partial K}{\partial x_j}(x,y)^2 \, \mathrm{d}y < \infty,$$

for all $j = 1, \ldots, n$. Clearly,

$$\int_{\mathbb{R}} \frac{\partial K}{\partial x_j}(x,y)^2 \, \mathrm{d}y \leq \sup_{y \in \mathbb{R}} \left(\frac{\partial K}{\partial x_j}(x,y) \right) \int_{\mathbb{R}} \left| \frac{\partial K}{\partial x_j}(x,y) \right| \, \mathrm{d}y. \tag{3.23}$$

Proposition 3.2.6. *For each $j = 1, \ldots n$,*

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}} \left| \frac{\partial K}{\partial x_j}(x,y) \right| \, \mathrm{d}y < \infty.$$

Proof. We first assume (and prove later) that we can differentiate under the integral sign, that is

$$\frac{\partial K}{\partial x_j}(x, y_1) = \int_{\mathbb{R}^{n-1}} \frac{\partial Q_1}{\partial x_j}(x,y) \, \mathrm{d}y_2 \ldots \mathrm{d}y_n. \tag{3.24}$$

By the Harish-Chandra formula (3.17), $Q_1(x,y)$ can be written as

$$\begin{aligned} Q_1(x,y) &= (2\pi)^{-n/2} c_n \int_{\mathcal{U}(n)} \Delta(y)^2 \exp\Big(-\frac{1}{2}\mathrm{Tr}(Y-UXU^\dagger)^2\Big)\,\mathrm{d}U \\ &= (2\pi)^{-n/2} c_n \int_{\mathcal{U}(n)} \Delta(y)^2 \exp\Big(-\frac{1}{2}\mathrm{Tr}(D_y-UD_xU^\dagger)^2\Big)\,\mathrm{d}U, \end{aligned}$$

where D_x, D_y are diagonal matrices with the eigenvalues of X and Y as its entries respectively. The second equality follows from the first due to the invariance of Haar measure on $\mathcal{U}(n)$. Observe that by the cyclic property of the trace and the fact that U is unitary, $\mathrm{Tr}(D_y - UD_xU^\dagger)^2 = \mathrm{Tr}(U^\dagger D_y U - D_x)^2$. Expanding the trace inside the exponential we have

$$\mathrm{Tr}(D_y - UD_xU^\dagger)^2 = \mathrm{Tr}\, D_y^2 + \mathrm{Tr}\, D_x^2 - 2\mathrm{Tr}\, D_x U^\dagger D_y U.$$

Therefore,

$$\frac{\partial Q_1}{\partial x_j}(x,y) = c_n' \int_{\mathcal{U}(n)} \Delta(y)^2 \big((U^\dagger D_y U)_{jj} - x_j\big) \exp\Big(-\frac{1}{2}\mathrm{Tr}(D_x - U^\dagger D_y U)^2\Big)\,\mathrm{d}U, \quad (3.25)$$

where $c_n' = (2\pi)^{-n/2} c_n$. For a Hermitian matrix H, one can check that $\mathrm{Tr} H^2 = \sum_{i=1}^n h_{ii}^2 + 2\sum_{i<j} |h_{ij}|^2$ and so $\mathrm{Tr}(D_x - U^\dagger D_y U)^2 = \sum_{i=1}^n \big(x_i - (U^\dagger D_y U)_{ii}\big)^2 + 2\sum_{i<j} \big|(U^\dagger D_y U)_{ij}\big|^2$. Then,

$$\begin{aligned} &\left|\frac{\partial Q_1}{\partial x_j}(x,y)\right| \\ &\quad \le C c_n' \int_{\mathcal{U}(n)} \Delta(y)^2 \exp\Big(-\frac{1}{4}\sum_{i=1}^n \big((U^\dagger D_y U)_{ii} - x_i\big)^2 - \frac{1}{2}\sum_{i<j} \big|(U^\dagger D_y U)_{ij}\big|^2\Big)\,\mathrm{d}U, \quad (3.26) \end{aligned}$$

where $C = 2\sup_{x\in\mathbb{R}} xe^{-x^2} = \sqrt{2/e}$. Hence,

$$\int_{\mathbb{R}} \left|\frac{\partial K}{\partial x_j}(x,y_1)\right| \mathrm{d}y_1 \le C c_n' \int_{\mathbb{R}^n} \int_{\mathcal{U}(n)} \Delta(y)^2 \exp\Big(-\frac{1}{4}\mathrm{Tr}(U^\dagger D_y U - D_x)^2\Big)\,\mathrm{d}U \prod_{i=1}^n \mathrm{d}y_i.$$

We can make a standard change of variables to the space of $n\times n$ Hermitian matrices $\mathcal{H}(n)$ by the rule $\mathrm{d}Y = Z_n \Delta(y)^2\,\mathrm{d}y\mathrm{d}U$ where $Z_n = c_n \pi^{n(n-1)/2}$ and $\mathrm{d}Y$ is the product of Lebesgue measures $\prod_{i\le j} \mathrm{d}y_{ij} \prod_{i<j} \mathrm{d}y_{ji}$. The right hand side of the previous display is then equal to

$$\begin{aligned} &2^{-n/2}\pi^{-n^2/2} \int_{\mathcal{H}(n)} e^{-\mathrm{Tr}(Y-D_x)^2/4}\,\mathrm{d}Y \\ &\qquad = 2^{-n/2}\pi^{-n^2/2} \int_{\mathbb{R}^{n^2}} \prod_{i=1}^n e^{-(y_{ii}-x_i)^2/4} \prod_{i<j} e^{-(y_{ij}^2+y_{ji}^2)/2}\,\mathrm{d}Y \le 2^{n^2/2}. \end{aligned}$$

It remains to justify the swapping of the derivative and the integral in (3.24) and (3.25). For this we shall use the following result from [Bil95, Theorem 16.8].

Proposition 3.2.7. *Let (Y, μ) be a measure space. Suppose that $f(x, y)$ is a continuous and integrable function of y for each $x \in I$, where I can be taken to be $\mathbb{R}$ and that for each $y \in Y$, $\frac{\partial f}{\partial x}(x, y)$ exists. If for each x^* there exists a function $g(x^*, y)$ integrable in y such that $\left|\frac{\partial f}{\partial x}(x, y)\right| \leq g(x^*, y)$ for all y and all x in some neighbourhood of x^*, then*

$$\frac{\partial}{\partial x} \int_Y f(x, y)\, \mu(\mathrm{d}y) = \int_Y \frac{\partial f}{\partial x}(x, y)\, \mu(\mathrm{d}y).$$

Thus, we need to show that $Q_1(x, y)$ satisfies the hypothesis of the above proposition. Since the function $x \mapsto p_n^*(t, x, y)/\Delta(x)\Delta(y)$ is smooth on $\mathbb{R}^n$, the same is true for $Q_t(x, y)$ so it remains to find a dominating function g.

Firstly, for (3.25), one can apply Proposition 3.2.7 with g equal to a constant since $e^{-\mathrm{Tr}(D_y - U D_x U^\dagger)^2/2} \leq 1$ and $\mathcal{U}(n)$ is compact. For (3.24), consider the interval $[x_j^* - h, x_j^* + h]$ around a fixed point $x_j^* \in \mathbb{R}$ where $h > 0$. Then for $x_j \in [x_j^* - h, x_j^* + h]$, we have

$$e^{-(y_j - x_j)^2/2} = e^{-y_j^2/2} e^{-x_j^2/2} e^{x_j y_j} \leq e^{-y_j^2/2} e^{(x_j^* + h)|y_j|} = e^{-(y_j - (x_j^* + h))^2/2} e^{(x_j^* + h)^2/2}.$$

Therefore, for such x_j, we have by the bounds (3.26) and (3.18) that

$$\begin{aligned}
\left|\frac{\partial Q_1}{\partial x_j}(x, y)\right| &\leq C c_n' \int_{\mathcal{U}(n)} \Delta(y)^2 \exp\Big(-\frac{1}{4}\mathrm{Tr}(U^\dagger D_y U - D_x)^2\Big)\, \mathrm{d}U \\
&\leq C c_n' \Delta(y)^2 \prod_{i \neq j} e^{-(y_i - x_i)^2/4} e^{-(y_j - (x_j^* + h))^2/4} e^{(x_j^* + h)^2/4} \\
&=: g(x^*, y),
\end{aligned}$$

and g is integrable over $\mathbb{R}^{n-1}$ with respect to $y_2, \ldots, y_n$ due to the Gaussian factor. Considering y_1, x_i, $i \neq j$ fixed and applying Proposition 3.2.7 with the above g proves (3.24) and hence completes the proof. $\square$

Proposition 3.2.8. *For all $j = 1, \ldots, n$*

$$\sup_{(x,y) \in \mathbb{R}^n \times \mathbb{R}} \frac{\partial K}{\partial x_j}(x, y) < \infty.$$

To prove this we shall use the following formula for the one point correlation function K. For $1 \leq N \leq n$ it was shown in [Joh01b, Proposition 2.3] that the N-point correlation function of Q_t is given by a determinant:

$$\frac{n!}{(n-N)!} \int_{\mathbb{R}^{n-N}} Q_t(x, y)\, \mathrm{d}y_{N+1} \ldots \mathrm{d}y_n = \det\left[\tilde{K}_t(x, y_i, y_j)\right]_{1 \leq i,j \leq N},$$

where

$$\tilde{K}_t(x, u, v) = \frac{1}{(2\pi i)^2 t} \int_\gamma \mathrm{d}z \int_{\Gamma_L} \mathrm{d}w\, e^{\frac{1}{2t}(w-v)^2 - \frac{1}{2t}(z-u)^2} \frac{1}{w - z} \prod_{j=1}^n \frac{w - x_j}{z - x_j} \tag{3.27}$$

where γ is a closed contour around the x_i's and $\Gamma_L : t \to L + it$, $t \in \mathbb{R}$ with $L \in \mathbb{R}$ large enough so that γ and Γ_L do not intersect. Then $K(x, y)$ is simply $\frac{(n-1)!}{n!}\tilde{K}_1(x, y, y)$. It is sometimes convenient to use the following alternate expression for $\tilde{K}_t$, see the equation below [Joh01b, equation (2.18)]:

$$\begin{aligned}\tilde{K}_t(x, u, v) = &-\frac{1}{(2\pi i)^2 t^2}\int_\gamma \mathrm{d}z \int_{\Gamma_L} \mathrm{d}w\, e^{\frac{1}{2t}(w-v)^2 - \frac{1}{2t}(z-u)^2}\frac{1}{w-z}\prod_{j=1}^n \frac{w-x_j}{z-x_j} \\ &\times \left[(w+z)(w-z) + uz - vw + t\sum_{j=1}^n \frac{x_j(w-z)}{(w-x_j)(z-x_j)}\right], \end{aligned} \tag{3.28}$$

with the same contours as before. Observe that the integral formulas (3.27) and (3.28) make clear the symmetry of $\tilde{K}_t$ with respect to the ordering of $x_1, \dots, x_n$ and that there are no issues if any of the x_i's coincide.

Lemma 3.2.9. *For all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$*

$$\frac{\partial K}{\partial x_j}(x, y) = \frac{1}{n}\int_\gamma \frac{\mathrm{d}z}{2\pi i}\int_{\Gamma_0} \frac{\mathrm{d}w}{2\pi i}\frac{e^{-(z-y)^2/2}e^{(w-y)^2/2}}{(z-x_j)^2}\prod_{i\neq j}\frac{w-x_i}{z-x_i}. \tag{3.29}$$

Proof. Since

$$\frac{\partial}{\partial x_j}\prod_{i=1}^n \frac{w-x_i}{z-x_i} = \frac{w-z}{(z-x_j)^2}\prod_{i\neq j}\frac{w-x_i}{z-x_i},$$

the derivative with respect to x_j of the integrand in the formula for $K(x, y)$ is equal to

$$f(x; z, w) := \frac{1}{n}\frac{e^{-(z-y)^2/2}e^{(w-y)^2/2}}{(z-x_j)^2}\prod_{i\neq j}\frac{w-x_i}{z-x_i}.$$

The rest of the proof is devoted to justifying the exchange of integral and derivative. Consider a bounded set B in the complex plane and let $x = (x_1, \dots, x_n)$ with the x_i's all lie on the real line in B. Let γ be a closed contour containing B and therefore also contains x, then there exist constants $d > 0$, $C > 0$ such that for all $z \in \gamma$, $|z - x_i| \geq d$ for all i and $|z| \leq C$. Moreover,

$$\left|\frac{w-x_i}{z-x_i}\right| = \left|1 + \frac{w-z}{z-x_i}\right| \leq 1 + \frac{|w| + |z|}{d}. \tag{3.30}$$

Therefore, for all $x \in B$ there is a constant b_n such that

$$\begin{aligned}|f(x; z, w)| &\leq \frac{b_n}{d^{n+1}}\sup_{z\in\gamma}|e^{-(z-y)^2/2}||e^{(w-y)^2/2}|\big((d+C)^{n-1} + |w|^{n-1}\big) \\ &=: g(z, w).\end{aligned}$$

The function g is integrable along the contours γ and Γ_L. Indeed,

$$\begin{aligned}\frac{b_n}{d^{n+1}}\int_\gamma \mathrm{d}z \int_{\Gamma_L} \mathrm{d}w \sup_{z\in\gamma}|e^{-(z-y)^2/2}||e^{(w-y)^2/2}|(d+C)^{n-1} \\ = \frac{b_n \mathrm{length}(\gamma)}{d^{n+1}}(d+C)^{n-1}\sup_{z\in\gamma}|e^{-(z-y)^2/2}|\int_{\Gamma_y}\mathrm{d}w\ |e^{(w-y)^2/2}|,\end{aligned}$$

where in the last line we have shifted the contour Γ_L to $\Gamma_y : t \to y+it$ by Cauchy's theorem. The integral with respect to w is just a Gaussian integral and integrates to a constant. The other term is treated in a similar fashion but the $\mathrm{d}w$ integral is instead equal to

$$\int_{\Gamma_y}\mathrm{d}w\ |w|^{n-1}|e^{(w-y)^2/2}| = \int_{\mathbb{R}}|y+it|^{n-1}e^{-t^2/2}\ \mathrm{d}t < \infty,$$

for each fixed $y \in \mathbb{R}$. Thus, by Proposition 3.2.7, we can differentiate under the integral to see that the derivative of $K(x,y)$ is given by

$$\frac{\partial K}{\partial x_j}(x,y) = \frac{1}{n}\int_\gamma \frac{\mathrm{d}z}{2\pi i}\int_{\Gamma_L}\frac{\mathrm{d}w}{2\pi i}\frac{e^{-(z-y)^2/2}e^{(w-y)^2/2}}{(z-x_j)^2}\prod_{i\neq j}\frac{w-x_i}{z-x_i}$$

Finally, by Cauchy's theorem we can shift the contour Γ_L so that $L=0$ since there is no longer a pole at $z=w$. □

Proof of Proposition 3.2.8. It is clear from the contour integral (3.29) that $\frac{\partial K}{\partial x_j}(x,y)$ is translation invariant in the sense that $\frac{\partial K}{\partial x_j}(x+h\mathbf{1}, y+h) = \frac{\partial K}{\partial x_j}(x,y)$ for all $h \in \mathbb{R}$. Hence, $\sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}}\frac{\partial K}{\partial x_j}(x,y)$ is equivalent to $\sup_{x\in\mathbb{R}^n}\frac{\partial K}{\partial x_j}(x,0)$ so we only need to bound the latter. Fix a constant $d>0$. By Cauchy's theorem, we can take γ to be the closed (rectangular) contour around $x_1,\dots,x_n$ composed of four parts γ_t, γ_b, γ_r and γ_l, where $\gamma_t : u \to -u+di$, $u\in[-R,R]$, $\gamma_b : u \to u-di$, $u \in [-R,R]$, $\gamma_r : v \to R+vi$, $v\in[-d,d]$, and $\gamma_l : v \to -R-vi$, $v \in [-d,d]$. $R := R(x)$ is chosen so that the minimum distance between the contour γ and the x_i's is at least d. We shall consider each parts of the contour separately. Denote the integral along the contour γ_t by $I(\gamma_t)$ and likewise for the others.

Since $|z-x_i|\geq d$ for all i and $z\in\gamma$, we have by (3.30) that

$$\prod_{i\neq j}\left|\frac{w-x_i}{z-x_i}\right| \leq \left(1+\frac{|w|+|z|}{d}\right)^{n-1} \leq \frac{2^{n-2}}{d^{n-1}}\big((d+|z|)^{n-1}+|w|^{n-1}\big).$$

On γ_r, $|z| = |R+vi| = (R^2+v^2)^{1/2} \leq (R^2+d^2)^{1/2}$ and

$$|e^{-z^2/2}| = |e^{-(R^2+2iRv-v^2)/2}| \leq e^{-R^2/2}e^{d^2/2}.$$

Therefore

$$
\begin{aligned}
|I(\gamma_r)| \leq{}& \frac{2^{n-2}}{d^{n+1}} \int_{\gamma_r} \frac{\mathrm{d}z}{2\pi} \left(d + (R^2 + d^2)^{1/2}\right)^{n-1} e^{-R^2/2} e^{d^2/2} \int_{\mathbb{R}} \frac{1}{2\pi} e^{-t^2/2}\, \mathrm{d}t \\
&+ \frac{2^{n-2}}{d^{n+1}} \int_{\gamma_r} \frac{\mathrm{d}z}{2\pi}\, e^{-R^2/2} e^{d^2/2} \int_{\mathbb{R}} \frac{1}{2\pi} |t|^{n-1} e^{-t^2/2}\, \mathrm{d}t \\
={}& \frac{2^{n-2}}{(2\pi)^{3/2} d^{n+1}} \mathrm{length}(\gamma_r) \left(d + (R^2 + d^2)^{1/2}\right)^{n-1} e^{-R^2/2} e^{d^2/2} \\
&+ \frac{C_n 2^{n-2}}{(2\pi)^{3/2} d^{n+1}} \mathrm{length}(\gamma_r) e^{-R^2/2} e^{d^2/2},
\end{aligned} \tag{3.31}
$$

where $\mathrm{length}(\gamma_r) = 2d$ and

$$
C_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |t|^{n-1} e^{-t^2/2}\, \mathrm{d}t = \begin{cases} (n-2)!! & \text{if } n \text{ odd} \\ 2^{n/2}\left(\frac{1}{2}(n-1)\right)! & \text{if } n \text{ even} \end{cases}, \quad n \geq 2. \tag{3.32}
$$

Due to the exponential term $e^{-R^2/2}$ we see that the two terms on the right hand side of (3.31) vanishes as $R \to \infty$ and hence the same is true for $I(\gamma_r)$. By symmetry, the same argument shows that $I(\gamma_l)$ also vanishes as $R \to \infty$. Thus, we can deform the contour γ to the two horizontal lines, $\gamma_+ : u \to -u + di$ and $\gamma_- : u \to u - di$, $u \in \mathbb{R}$.

On γ_+, $|z| = (u^2 + d^2)^{1/2}$ and $|e^{-z^2/2}| = |e^{-(-u+di)^2/2}| \leq e^{-u^2/2} e^{d^2/2}$. Hence, in a similar fashion as above, we have

$$
\begin{aligned}
|I(\gamma_+)| \leq{}& \frac{2^{n-2}}{2\pi d^{n+1}} e^{d^2/2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(d + (u^2 + d^2)^{1/2}\right)^{n-1} e^{-u^2/2}\, \mathrm{d}u \\
&+ \frac{C_n 2^{n-2}}{2\pi d^{n+1}} e^{d^2/2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-u^2/2} \mathrm{d}u \\
={}& \frac{2^{n-2}}{2\pi d^{n+1}} e^{d^2/2} \left(C_n' + C_n\right),
\end{aligned}
$$

where

$$
\begin{aligned}
C_n' ={}& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(d + (u^2 + d^2)^{1/2}\right)^{n-1} e^{-u^2/2}\, \mathrm{d}u \\
\leq{}& \frac{2^{n-2}}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(d^{n-1} + (u^2 + d^2)^{(n-1)/2}\right) e^{-u^2/2}\, \mathrm{d}u \\
\leq{}& \frac{2^{n-2}}{\sqrt{2\pi}} \int_{\mathbb{R}} d^{n-1} e^{-u^2/2} + 2^{(n-3)/2} (u^{n-1} + d^{n-1}) e^{-u^2/2}\, \mathrm{d}u \\
={}& 2^{n-2} d^{n-1} (1 + 2^{(n-3)/2}) + \frac{2^{n-2} 2^{(n-3)/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} u^{n-1} e^{-u^2/2}\, \mathrm{d}u,
\end{aligned}
$$

and the integral on the last line is equal to zero if n is even and equal to $(n-2)!!$ if n is odd. By symmetry, the same bound applies for $I(\gamma_-)$ and hence we have shown that there exists a constant C depending only on n and d and is independent of x such that

$$\sup_{x\in\mathbb{R}^n}\left|\frac{\partial K}{\partial x_j}(x,0)\right| \le \sup_{x\in\mathbb{R}^n}\big(|I(\gamma_+)|+|I(\gamma_-)|\big)\le C,$$

as required. $\square$

We now turn our attention to showing (3.21). Observe that

$$\int_{\mathbb{R}} K(x,y)^2 \,\mathrm{d}y \le \sup_{y\in\mathbb{R}} K(x,y)\int_{\mathbb{R}} K(x,y)\,\mathrm{d}y = n!\sup_{y\in\mathbb{R}} K(x,y), \tag{3.33}$$

since $\int_{W_n} Q_1(x,y)\,\mathrm{d}y = 1$ for all x. So it suffices to show that $\sup_{x,y} K(x,y)$ is bounded or equivalently by the translation invariance of K which follows from its integral representation that $\sup_{x\in\mathbb{R}^n} K(x,0)$ is bounded.

Lemma 3.2.10.

$$\sup_{x\in\mathbb{R}^n} K(x,y) = \sup_{x\in\mathbb{R}^n} K(x,0) < \infty.$$

Proof. It is convenient to use the contour integral formula (3.28) instead. Notice that there is no longer a pole at $w=z$ and so we can deform the contour Γ_L so that $L=0$. Let γ be the contour in the proof of Proposition 3.2.8 comprising of four parts, γ_r, γ_l, γ_t and γ_b. It can be shown in the same manner as in the proof of Proposition 3.2.8 that the contributions from γ_r and γ_l vanishes at infinity in the direction of the real axis and so we can deform the contour γ to the two horizontal lines, $\gamma_+ : u\to -u+di$ and $\gamma_- : u\to u-di$, $u\in\mathbb{R}$ for a fixed $d>0$. We then have

$$\begin{aligned} nK(x,0) &= -\frac{1}{(2\pi i)^2}\int_{\gamma_+\cup\gamma_-}\mathrm{d}z\int_{\Gamma_0}\mathrm{d}w\; e^{-z^2/2}e^{w^2/2}(w+z)\prod_{j=1}^n\frac{w-x_j}{z-x_j}\\ &\quad + -\frac{1}{(2\pi i)^2}\int_{\gamma_+\cup\gamma_-}\mathrm{d}z\int_{\Gamma_0}\mathrm{d}w\; e^{-z^2/2}e^{w^2/2}\prod_{j=1}^n\frac{w-x_j}{z-x_j}\sum_{j=1}^n\frac{x_j}{(w-x_j)(z-x_j)}\\ &=: I_1+I_2. \end{aligned} \tag{3.34}$$

Denote the contribution from γ_+ by $I_j(\gamma_+)$, $j=1,2$ and likewise for γ_-. Note that on γ_+, $|z|^2=(u^2+d^2)$, $|e^{-z^2/2}|\le e^{-u^2/2}e^{d^2/2}$ and $|z-x_j|\ge d$ for all j. Hence, by (3.30) we have in a similar manner to the proof of Proposition 3.2.8 that

$$\begin{aligned} |I_1(\gamma_+)| &\le \frac{e^{d^2/2}}{4\pi^2}\int_{\mathbb{R}}\mathrm{d}z\int_{\mathbb{R}}\mathrm{d}t\; e^{-u^2/2}e^{-t^2/2}(|t|+(u^2+d^2)^{1/2})\left(1+\frac{|t|+(u^2+d^2)^{1/2}}{d}\right)^n\\ &\le C_{d,n}, \end{aligned} \tag{3.35}$$

for some constant $C_{d,n}$. By symmetry $I_1(\gamma_-)$ is bounded by the same constant.

It remains to bound I_2. Observe that

$$\left|\prod_{j=1}^{n} \frac{w-x_j}{z-x_j} \sum_{k=1}^{n} \frac{x_k}{(w-x_k)(z-x_k)}\right| \leq \prod_{j=1}^{n} \left|\frac{w-x_j}{z-x_j}\right| \sum_{k=1}^{n} \frac{1}{|z-x_k|} \leq \frac{n}{d}\left(1+\frac{|w|+|z|}{d}\right)^n.$$

Thus in the same way as above, both $|I_2(\gamma_+)|$ and $|I_2(\gamma_-)|$ are bounded by some constant $C'_{d,n}$. Combining this with (3.34) and (3.35) shows that there exists a constant C independent of x and depending only on n and d such that

$$\sup_{x\in\mathbb{R}^n} K(x,0) \leq C,$$

which completes the proof. □

Proof of Theorem 3.2.5(a). Lemma 3.2.10, Proposition 3.2.6 Proposition 3.2.8 and (3.23), (3.33) together imply that (3.21) and (3.22) are bounded. This in turn shows that the assumption of Lemma 2.2.5 is satisfied and the result follows. □

Lemma 3.2.11. *There exists a constant $C_4 > 0$ depending only on n such that for all $t > 0$ and $x \in \mathbb{R}^n$,*

$$\int_{\mathbb{R}} K_t(x,y)^2 \,\mathrm{d}y \leq C_4 t^{-1/2}.$$

Proof. By the scaling property of Q_t and a change of variables

$$\int_{\mathbb{R}} K_t(x,y)^2 \,\mathrm{d}y = t^{-1/2} \int_{\mathbb{R}} K_1(xt^{-1/2},y')^2 \,\mathrm{d}y'.$$

By Lemma 3.2.10 and (3.33), the latter integral for each fixed n is bounded uniformly in x which gives the desired result. □

Proof of Theorem 3.2.5(b). Let $t = u+h$ where $h > 0$, then we need to estimate

$$\int_0^u \int_{\mathbb{R}} \big(K_{s+h}(x,y) - K_s(x,y)\big)^2 \,\mathrm{d}y\mathrm{d}s.$$

Making the change of variable $s = hs'$, $y = \sqrt{h}y'$ and using the scaling property (3.19) of Q_t, the above is bounded by

$$h^{1/2} \int_0^\infty \int_{\mathbb{R}} \big(K_{s'+1}(x/\sqrt{h},y') - K_{s'}(x/\sqrt{h},y')\big)^2 \,\mathrm{d}y'\mathrm{d}s',$$

and hence it suffices to show that

$$\int_0^\infty \int_{\mathbb{R}} \big(K_{s+1}(x,y) - K_s(x,y)\big)^2 \,\mathrm{d}y\mathrm{d}s < \infty$$

uniformly for $x \in \mathbb{R}^n$. Firstly, by Lemma 3.2.11

$$\int_0^1 \int_{\mathbb{R}} \left(K_{s+1}(x,y) - K_s(x,y)\right)^2 \,\mathrm{d}y\mathrm{d}s \leq 2\int_0^1 \int_{\mathbb{R}} K_{s+1}(x,y)^2 + K_s(x,y)^2 \,\mathrm{d}y\mathrm{d}s < \infty.$$

On the other hand

$$\int_1^\infty \int_{\mathbb{R}} \left(K_{s+1}(x,y) - K_s(x,y)\right)^2 \,\mathrm{d}y\mathrm{d}s = \int_1^\infty \int_{\mathbb{R}} \left(\int_s^{s+1} \frac{\partial K_r}{\partial r}(x,y)\,\mathrm{d}r\right)^2 \mathrm{d}y\mathrm{d}s,$$

and thus in the same way as in Proposition 3.2.6 we need to estimate the derivative of Q_t. Using the Harish-Chandra formula and denoting $A_U = (D_y - UD_xU^\dagger)^2$ we see that

$$\begin{aligned} \frac{\partial Q_r}{\partial r}(x,y) &= c_n r^{-n^2/2}\Delta(y)^2 \int_{\mathcal{U}(n)} e^{-\mathrm{Tr}A_U/2r}\left(\frac{\mathrm{Tr}A_U}{2r^2} - \frac{n^2}{2r}\right)\mathrm{d}U \\ &\leq \frac{C}{r} c_n r^{-n^2/2}\Delta(y)^2 \int_{\mathcal{U}(n)} e^{-\mathrm{Tr}A_U/4r}\,\mathrm{d}U \\ &= \frac{C}{r} Q_{2r}(x,y), \end{aligned} \tag{3.36}$$

for some constant $C > 0$ depending only on n. We assume for now that $\frac{\partial K_r}{\partial r}(x,y_1) = \int_{\mathbb{R}^{n-1}} \frac{\partial Q_r}{\partial r}(x,y) \prod_{i=2}^n \mathrm{d}y_i$ then the above shows that $\frac{\partial K_r}{\partial r}(x,y_1) \leq \frac{C}{r} K_{2r}(x,y_1)$. Therefore, by Minkowski's integral inequality and Lemma 3.2.11

$$\begin{aligned} \left(\int_{\mathbb{R}}\left(\int_s^{s+1} \frac{\partial K_r}{\partial r}(x,y)\,\mathrm{d}r\right)^2 \mathrm{d}y\right)^{1/2} &\leq C\left(\int_{\mathbb{R}}\left(\int_s^{s+1} \frac{1}{r}K_{2r}(x,y)\,\mathrm{d}r\right)^2 \mathrm{d}y\right)^{1/2} \\ &\leq C\int_s^{s+1} \frac{1}{r}\left(\int_{\mathbb{R}} K_{2r}(x,y)^2\,\mathrm{d}y\right)^{1/2} \mathrm{d}r \\ &\leq C' s^{-5/4}, \end{aligned}$$

for a constant $C' := C'(n) > 0$. Consequently,

$$\int_1^\infty \int_{\mathbb{R}} \left(K_{s+1}(x,y) - K_s(x,y)\right)^2 \,\mathrm{d}y\mathrm{d}s \leq C'^2 \int_1^\infty s^{-5/2}\,\mathrm{d}s < \infty.$$

It remains to justify the interchange of the integral and derivative in the estimate of $\frac{\partial K_r}{\partial r}$. Fix $r_* > 0$ then by the above estimate and using the bound (3.18) we have for $r \in [r_*/2, 2r_*]$

$$\begin{aligned} \frac{\partial Q_r}{\partial r}(x,y) &\leq C c_n r^{-n^2/2-1}\Delta(y)^2 \int_{\mathcal{U}(n)} e^{-\mathrm{Tr}A_U/4r}\,\mathrm{d}U \\ &\leq C' c_n r_*^{-n^2/2-1}\Delta(y)^2 \prod_{i=1}^n e^{-(x_i-y_i)^2/8r_*} \\ &=: g(r_*,x,y). \end{aligned}$$

The function g is integrable over $\mathbb{R}^{n-1}$ with respect to $y_2,\ldots,y_n$ and so an application of

Proposition 3.2.7 completes the argument.

Finally, by Lemma 3.2.11 we have

$$\int_u^t \int_{\mathbb{R}} K_s(x,y)^2 \, \mathrm{d}y\mathrm{d}s \le C_4 \int_u^t s^{-1/2} \, \mathrm{d}s \le 2C_4|t-u|^{1/2}.$$

This completes the entire proof of the theorem. □

3.3 Existence, Uniqueness and Moment Estimates

3.3.1 Bounded Initial Data

We now prove the existence, uniqueness and moment estimates part of Theorem 3.1.2(a). The proof of continuity will be delayed to Section 3.4. In the sequel constants will generally be denoted by c, C or K and possibly adorned with primes or subscripts. They may differ from line to line and their dependence if any will always be specified. However, C_i, $1 \le i \le 4$ will always mean the constants in Theorem 3.2.5 and Lemma 3.2.11. $T > 0$ will always denote the finite time horizon.

Proof of existence, uniqueness and moment estimates of Theorem 3.1.2(a). The proof is by a Picard iteration argument. Throughout the proof, we fix an arbitrary integer $p \ge 2$. For $(t,y) \in (0,\infty) \times \mathbb{R}^n$ define $m^0(t,y) := J_n(t,y)$ where J_n was defined in (3.13) and for $k \ge 1$, let

$$\begin{aligned} m^k(t,y) &= m^0(t,y) + A_n \int_0^t \int_{\mathbb{R}^n} Q_{t-s}(y,y') m^{k-1}(s,y') \, \mathrm{d}y'_* \, W(\mathrm{d}s, \mathrm{d}y'_1) \\ &=: m^0(t,y) + I^k(t,y). \end{aligned} \tag{3.37}$$

We first show that each of the stochastic integrals above are well defined, that is for all $(t,y) \in (0,\infty) \times \mathbb{R}^n$, the random field $\big(f_k(s,y), (s,y) \in (0,t) \times \mathbb{R}\big)$ defined by $f_k(s,y'_1) := \int_{\mathbb{R}^{n-1}} Q_{t-s}(y,y') m^k(s,y') \, \mathrm{d}y'_*$ is in $\mathscr{P}_2$ for all $k \ge 0$.

Fix $(t,y) \in (0,\infty) \times \mathbb{R}^n$ and consider $f_0(s,y'_1) = \int_{\mathbb{R}^{n-1}} Q_{t-s}(y,y') m^0(s,y') \, \mathrm{d}y'_*$. We need to show that m^0 satisfies the three assumptions of Proposition 3.2.4. Since the initial data g is $\mathscr{F}_0$-measurable, m^0 is adapted to the filtration $(\mathscr{F}_t)_{t\ge 0}$. By assumption on g, $\sup_{y\in\mathbb{R}^n} \|g(y)\|_p \le K_{p,g} < \infty$ and hence by Minkowski's integral inequality, we have for all $t > 0$

$$\begin{aligned} \|m^0(t,y)\|_p &\le \frac{1}{n!} \int_{\mathbb{R}^n} \|g(y')\|_p Q_t(y,y') \, \mathrm{d}y' \\ &\le \sup_{y\in\mathbb{R}^n} \|g(y)\|_p \frac{1}{n!} \int_{\mathbb{R}^n} Q_t(y,y') \, \mathrm{d}y' \\ &\le K_{p,g}. \end{aligned} \tag{3.38}$$

Therefore, $\|m^0(s,y)\|_p^2$ is bounded by $K_{p,g}^2$ uniformly for $(s,y) \in [0,\infty) \times \mathbb{R}^n$. By Lemma

3.4.1 below, $(s,y') \mapsto m^0(s,y')$ is $L^2(\Omega)$-continuous over $(0,t)\times\mathbb{R}^n$ and so Proposition 3.2.4 implies that $f_0 \in \mathscr{P}_2$ and

$$\int_0^t \int_{\mathbb{R}^n} Q_{t-s}(y,y')m^0(s,y')\,\mathrm{d}y'_*\,W(\mathrm{d}s,\mathrm{d}y'_1),$$

is a well-defined Walsh integral. Consequently, the random field $\big(m^1(t,y) = m^0(t,y) + I^1(t,y), (t,y)\in(0,\infty)\times\mathbb{R}^n\big)$ is well defined.

We wish to show that the sequence $\{m^k(t,y)\}_{k\geq 0}$ is Cauchy in $L^p(\Omega)$. To this end, let $d_k(t,y) := \|m^{k+1}(t,y) - m^k(t,y)\|_p$. By Lemma 3.2.3, Lemma 3.2.11 and (3.38), we have for all $(t,y)\in(0,\infty)\times\mathbb{R}^n$,

$$\begin{aligned}
d_0(t,y)^2 &\leq A_n^2 c_p^2 \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} Q_{t-s}(y,y')\|m^0(s,y')\|_p\,\mathrm{d}y'_* \right)^2 \mathrm{d}y'_1\mathrm{d}s \\
&\leq 2K_{p,g}^2 C_4 A_n^2 c_p^2 \sqrt{t} \\
&= K_{p,g}^2 C_4 A_n^2 c_p^2 \sqrt{\pi}\frac{\sqrt{t}}{\Gamma\big(\frac{3}{2}\big)},
\end{aligned}$$

where C_4 is the constant in Lemma 3.2.11 and $\Gamma(3/2) = \sqrt{\pi}/2$.

Now assume that for all $0 \leq l \leq k$, $\big(m^l(t,y), (t,y)\in(0,\infty)\times\mathbb{R}^n\big)$ is well defined and satisfies

(i) m^l is adapted,

(ii) $(s,y)\mapsto m^l(s,y)$ is $L^2(\Omega)$-continuous on $(0,t)\times\mathbb{R}^n$ for all $t>0$,

(iii) for all $(t,y)\in(0,\infty)\times\mathbb{R}^n$ and $0\leq l\leq k-1$

$$d_l(t,y)^2 \leq K_{p,g}^2 (C_4 A_n^2 c_p^2 \sqrt{\pi})^{l+1} \frac{t^{(l+1)/2}}{\Gamma\big(\frac{l+1}{2}+1\big)}.$$

We want to show that the same is true for m^{k+1} and d_k. Let $(t,y)\in(0,\infty)\times\mathbb{R}^n$. Observe that $m^k(t,y) = m^0(t,y) + \sum_{l=1}^k m^l(t,y) - m^{l-1}(t,y)$, and so to bound the pth moments of m^k it suffices to bound each of the d_l's, $0\leq l\leq k-1$. Indeed, by property (iii) and (3.38), we have

$$\begin{aligned}
\|m^k(t,y)\|_p^2 &\leq 2\|m^0(t,y)\|_p^2 + \sum_{l=1}^k 2^l d_{l-1}(t,y)^2 \\
&\leq 2K_{p,g}^2 \sum_{l=0}^k (C_4 A_n^2 c_p^2 \sqrt{\pi})^l \frac{t^{l/2}}{\Gamma\big(\frac{l}{2}+1\big)}, \qquad (3.39)
\end{aligned}$$

which shows that $\sup_{(s,y)\in[0,t]\times\mathbb{R}^n} \|m^k(s,y)\|_2 < \infty$. This and the induction hypothesis shows that m^k satisfies all three assumptions of Proposition 3.2.4 and so $f_k \in \mathscr{P}_2$ and

$$I^{k+1}(t,y) = A_n \int_0^t \int_{\mathbb{R}^{n-1}} Q_{t-s}(y,y') m^k(s,y')\, \mathrm{d}y'_*\, W(\mathrm{d}s, \mathrm{d}y'_1),$$

is a well-defined Walsh integral for all $(t,y) \in (0,\infty) \times \mathbb{R}^n$. Moreover, it is adapted and so $m^{k+1} = m^0 + I^{k+1}$ is also adapted. We need to check the $L^2(\Omega)$-continuity of I^{k+1}. By Theorem 3.2.5 we have for all $0 \le r \le u \le t$ and $y,\, z \in \mathbb{R}^n$ that

$$\begin{aligned}
&\|I^{k+1}(u,y) - I^{k+1}(r,z)\|_2^2 \\
&\qquad \le 2A_n^2 \int_0^r \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} \big(Q_{u-s}(y,y') - Q_{r-s}(z,y')\big) \|m^k(s,y'\|_2\, \mathrm{d}y'_* \right)^2 \mathrm{d}y'_1 \mathrm{d}s \\
&\qquad\quad + 2A_n^2 \int_r^u \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} Q_{u-s}(y,y') \|m^k(s,y')\|_2\, \mathrm{d}y'_* \right)^2 \mathrm{d}y'_1 \mathrm{d}s \\
&\qquad \le 2A_n^2 (C_1 + C_2 + C_3) \sup_{(s,y') \in [0,t] \times \mathbb{R}^n} \|m^k(s,y')\|_2^2 \big(|y-z| + |u-r|^{1/2}\big),
\end{aligned}$$

which proves the $L^2(\Omega)$-continuity of m^{k+1} on $(0,t) \times \mathbb{R}^n$.

For the bound on d_k, we use Lemmata 3.2.3 and 3.2.11 and the induction hypothesis to obtain

$$\begin{aligned}
d_k(t,y)^2 &\le K_{p,g}^2 (C_4 A_n^2 c_p^2)^{k+1} \pi^{k/2} \int_0^t \frac{s^{k/2}}{\Gamma\big(\frac{k}{2}+1\big)} (t-s)^{-1/2}\, \mathrm{d}s \\
&= K_{p,g}^2 (C_4 A_n^2 c_p^2 \sqrt{\pi})^{k+1} \frac{t^{(k+1)/2}}{\Gamma\big(\frac{k+1}{2}+1\big)},
\end{aligned} \tag{3.40}$$

where we have used the Euler Beta integral (2.21) and the fact that $\Gamma(1/2) = \sqrt{\pi}$ to evaluate the time integral. It follows that the bound (3.39) holds with k replaced with $k+1$ and that $\sup_{(s,y) \in [0,t] \times \mathbb{R}^n} \|m^{k+1}(s,y)\|_2 < \infty$. Hence, m^{k+1} satisfies all the assumptions of Proposition 3.2.4 and therefore $f_{k+1} \in \mathscr{P}_2$.

Thus, by induction we conclude that for all integers k, the random field $\big(m^k(t,y) = m^0(t,y) + I^k(t,y), (t,y) \in (0,\infty) \times \mathbb{R}^n\big)$ is well defined and satisfies properties (i), (ii) and (iii) listed above.

We now show that the sequence $\{m^k(t,y)\}_{k \ge 0}$ is Cauchy in $L^p(\Omega)$. This follows from the fact that for any $T > 0$

$$\sup_{(t,y) \in [0,T] \times \mathbb{R}^n} \sum_{k=0}^{\infty} d_k(t,y) < \infty,$$

which is a consequence of property (iii), the ratio test and asymptotics of ratios of Gamma functions in the same way as in Chapter 2. We conclude that there exist a random field which we denote by $M_n(t,y)$ such that $m^k(t,y) \to M_n(t,y)$ as $k \to \infty$ in $L^p(\Omega)$ and almost surely for a subsequence uniformly in $y \in \mathbb{R}^n$ and $t \in [0,T]$.

Since each m^k is adapted, M_n is also adapted. The $L^2(\Omega)$-continuity of M_n is inherited from that of m^k since the convergence is uniform on $[0,T] \times \mathbb{R}^n$ for all $T > 0$. Now take $k \to \infty$ on both sides of (3.39) then using (2.30) with $x = 2C_4 A_n^2 c_p^2 \sqrt{\pi} t^{1/2}$ gives the bound (3.14) in the statement of the theorem. Thus, by Proposition 3.2.4, for all $(t,y) \in (0,\infty) \times \mathbb{R}^n$ the random field f defined by $f(s,y_1') = \int_{\mathbb{R}^{n-1}} Q_{t-s}(y,y') M_n(s,y')\, \mathrm{d}y_*'$ for $(s,y_1') \in (0,t) \times \mathbb{R}$ is in $\mathscr{P}_2$ and the stochastic integral

$$I_n(t,y) = \int_0^t \int_{\mathbb{R}^n} Q_{t-s}(y, y' M_n(s,y')\, \mathrm{d}y_*'\, W(\mathrm{d}s, \mathrm{d}y_1'),$$

is well defined.

It remains to show that the limit $M_n(t,y)$ solves (3.13). Fix $(t,y) \in (0,\infty) \times \mathbb{R}^n$. By definition, $m^k(t,y) = m^0(t,y) + I^k(t,y)$ where the left hand side converges in $L^p(\Omega)$ and almost surely for a subsequence to $M_n(t,y)$. For the right hand side we have by the uniform convergence $L^p(\Omega)$ of m^k that

$$\begin{aligned} \|I^k(t,y) - I_n(t,y)\|_p^2 &\le 2\sqrt{t} A_n^2 c_p^2 \sup_{(s,y') \in [0,t] \times \mathbb{R}^n} \|m^k(s,y') - M_n(s,y')\|_p^2 \\ &\to 0 \quad \text{as } k \to \infty. \end{aligned}$$

Therefore, we have $L^p(\Omega)$ convergence of $I^k(t,y)$ to $I_n(t,y)$ and hence almost sure convergence for a subsequence to the same limit. The limit of both sides of $m^k(t,y) = m^0(t,y) + I^k(t,y)$ must be equal almost surely and so we have shown that for all $(t,y) \in (0,\infty) \times \mathbb{R}^n$, $M_n(t,y)$ satisfies (3.13) almost surely. This proves existence.

For uniqueness, suppose that $M(t,y)$ and $N(t,y)$ are both solutions to (3.13) with the same initial data g and let $d(t,y) = \|M(t,y) - N(t,y)\|_p$ then by a similar calculation as for existence we have

$$d(t,y)^2 \le \sup_{(s,y) \in [0,t] \times \mathbb{R}^n} d(s,y)^2\, (C_4 A_n^2 c_p^2 \sqrt{\pi})^n \frac{t^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}, \tag{3.41}$$

which converges to 0 as $n \to \infty$. Therefore, $d \equiv 0$ and so for all (t,y), $M(t,y) = N(t,y)$ almost surely i.e. M and N are versions of each other. This proves uniqueness. □

3.3.2 Delta Initial Data

Proof of existence, uniqueness and moment estimates of Theorem 3.1.2(b). Fix an integer $p \ge 2$. We first show that if solutions to (3.11) exists then it must be unique. Suppose $M(t,x,y)$ and $N(t,x,y)$ are two solutions to (3.11) and let $d(t,x,y) = \|M(t,x,y) - N(t,x,y)\|_p$. By linearity of the equation (3.11), $M(t,x,y) - N(t,x,y)$ is a solution to (3.13) with zero initial condition i.e. $M(t,x,y) - N(t,x,y) = M_n^g(t,y)$ with $g \equiv 0$. Then by (3.14), $\sup_{x,y \in \mathbb{R}^n} d(t,x,y)^2$ is a bounded function of $t \in [0,T]$ for any $T > 0$. The bound (3.41) applies to $d(t,x,y)^2$ which shows that $M(t,x,y) = N(t,x,y)$ almost surely for all (t,x,y). This proves uniqueness.

We now prove existence. We shall show that $M_n(t,x,y)$ defined by (3.9) satisfies equation (3.11) for all $(t,x,y) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$. Recall that $M_n(t,x,y)$ is well defined on the boundary of the Weyl chamber and it is symmetric under permutations of both its space variables, hence we can extend it to a function on $\mathbb{R}^n \times \mathbb{R}^n$. Similarly we also extend $Q_{t-s}(x,y)$ to the whole of $\mathbb{R}^n \times \mathbb{R}^n$. Substituting the chaos expansion of M_n into the stochastic integral term of (3.11), using the expression for the correlation function R_k (3.16) and the stochastic Fubini's theorem [Kho09, Theorem 5.30], we have bearing in mind that we can interchange the summation and integral because the series is convergent in $L^2(\Omega)$ that

$$
\begin{aligned}
&A_n \int_0^t \int_{\mathbb{R}^n} Q_{t-s_1}(y,y^1) M_n(s_1,x,y^1)\, \mathrm{d}y_*^1 W(\mathrm{d}s_1, \mathrm{d}y_1^1) \\
&= A_n \int_0^t \int_{\mathbb{R}^n} \frac{Q_{t-s_1}(y,y^1) p_n^*(s_1,x,y^1)}{\Delta(x)\Delta(y^1)}\, \mathrm{d}y_*^1\, W(\mathrm{d}s_1,\mathrm{d}y_1^1) \\
&\quad + A_n^{k+1} \int_0^t \int_{\mathbb{R}^n} \frac{p_n^*(t-s_1,y,y^1)}{\Delta(x)\Delta(y)} \sum_{k=1}^{\infty} \int_{\Delta_k(s_1)} \int_{(\mathbb{R}^n)^k} \prod_{i=2}^{k+1} p_n^*(s_{i-1}-s_i, y^{i-1}, y^i) \\
&\qquad \times p_n^*(s_{k+1}, y^{k+1}, x) \prod_{i=2}^{k+1} \mathrm{d}y_*^i\, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y})\, \mathrm{d}y_*^1\, W(\mathrm{d}s_1, \mathrm{d}y_1^1) \\
&= \frac{p_n^*(t,x,y)}{\Delta(x)\Delta(y)} \int_0^t \int_{\mathbb{R}} R_1(s_1, y_1^1; t,x,y)\, W(\mathrm{d}s_1, \mathrm{d}y_1^1) \\
&\quad + \sum_{k=1}^{\infty} \int_{\Delta_{k+1}(t)} \int_{(\mathbb{R}^n)^{k+1}} A_n^{k+1} \frac{p_n^*(t-s_1,y,y^1)}{\Delta(x)\Delta(y)} \prod_{i=2}^{k+1} p_n^*(s_{i-1}-s_i, y^{i-1}, y^i) \\
&\qquad \times p_n^*(s_{k+1}, y^{k+1}, x) \prod_{i=1}^{k+1} \mathrm{d}y_*^i\, W^{\otimes k+1}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) \\
&= \frac{p_n^*(t,x,y)}{\Delta(x)\Delta(y)} \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s},\mathbf{y};t,x,y)\, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}),
\end{aligned}
$$

where the last equality follows by a relabelling of the indices. Thus, the right hand side of (3.11) after the substitution is equal to

$$
\frac{p_n^*(t,x,y)}{\Delta(x)\Delta(y)} \left(1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s},\mathbf{y};t,x,y)\, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) \right),
$$

which is the definition of $M_n(t,x,y)$ as required.

It remains to estimate the pth moments of $M_n(t,x,y)$. The approach is to construct an approximating sequence to M_n and estimate the moments of each term of the sequence and take limits. The natural candidate for the approximating sequence is the following: for each $(t,x,y) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$, let $m^0(t,x,y) := J_n(t,x,y)$ where J_n was defined in (3.11) and for $k \geq 1$ define

$$
m^k(t,x,y) = m^0(t,x,y) \left(1 + \sum_{l=1}^{k} \int_{\Delta_l(t)} \int_{\mathbb{R}^l} R_l(\mathbf{s},\mathbf{y};t,x,y)\, W^{\otimes l}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) \right).
$$

In other words, $m^k(t,x,y)$ is the kth partial sum of the chaos expansion for $M_n(t,x,y)$. Let

$$d_{k-1}(t,x,y) = m^0(t,x,y) \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s},\mathbf{y};t,x,y)\, W^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{y}),$$

then by Lemma 2.2.12

$$\|d_{k-1}(t,x,y)\|_p^2 \le c_p^{2k} m^0(t,x,y)^2 \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s},\mathbf{y};t,x,y)^2 \,\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{s}. \tag{3.42}$$

Therefore,

$$\begin{aligned} \|m^k(t,x,y)\|_p^2 &\le 2m^0(t,x,y)^2 + \sum_{l=1}^k 2^l \|d_{l-1}(t,x,y)\|_p^2 \\ &\le 2m^0(t,x,y)^2 \bigg(1 + \sum_{l=1}^k (2c_p^2)^l \int_{\Delta_l(t)} \int_{\mathbb{R}^l} R_l(\mathbf{s},\mathbf{y};t,x,y)^2 \,\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{s}\bigg). \end{aligned}$$

Each term in the sum above is equal to $(2c_p^2)^l \mathbb{E}_{x,y;t}^{X,Y}\big[\big(\sum_{i,j=1}^n L_t(X^i - Y^j)\big)^l\big]/l!$ by Lemma 3.2.1 where $X = (X^1,\dots,X^n)$, $Y = (Y^1,\dots,Y^n)$ are independent copies of a collection of n non-intersecting Brownian bridges which start at x in time 0 and end at y in time t. Letting $k \to \infty$ we have for all $(t,x,y) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$

$$\lim_{k\to\infty} \|m^k(t,x,y)\|_p^2 \le 2m^0(t,x,y)^2 \mathbb{E}_{x,y;t}^{X,Y}\Big[\exp\Big(2c_p^2 \sum_{i,j=1}^n L_t(X^i - Y^j)\Big)\Big]. \tag{3.43}$$

For each $t > 0$, Lemma 3.2.2 shows that the right hand side of the above is bounded uniformly in x, $y \in \mathbb{R}^n$ for any $p \ge 2$. By Cauchy–Schwarz inequality

$$\|m^k(t,x,y) - m^{k'}(t,x,y)\|_p^p \le \|m^k(t,x,y) - m^{k'}(t,x,y)\|_2 \|m^k(t,x,y) - m^{k'}(t,x,y)\|_{2(p-1)}^{p-1},$$

which converges to 0 as k, $k' \to \infty$ by the $L^2(\Omega)$ convergence of m^k and the moment bound (3.43). Therefore, $m^k(t,x,y)$ also converges to $M_n(t,x,y)$ in $L^p(\Omega)$ and we can replace the left hand side of (3.43) with $\|M_n(t,x,y)\|_p^2$. This completes the proof of existence, uniqueness and moment estimates. □

3.4 Continuity

3.4.1 Bounded Initial Data

We now prove the Hölder continuity of the solution to (3.13) by verifying the assumptions of Kolmogorov's continuity criterion, Theorem 2.4.1. We first estimate the increments of $J_n(t,y) = \frac{1}{n!}\int_{\mathbb{R}^n} g(y')Q_t(y,y')\,\mathrm{d}y'$ where g satisfies the bound $\sup_{y\in\mathbb{R}^n} \|g(y)\|_p \le K_{p,g}$.

Lemma 3.4.1. *Let $M > 1$ and $p \ge 2$. There exist constants $K_i := K_i(g,M,n,p) > 0$,*

$i = 1, 2$ *such that for all* t, $t' \in [1/M, M]$ *and* y, $y' \in \mathbb{R}^n$

$$\|J_n(t, y) - J_n(t', y)\|_p \leq K_1 |t - t'|,$$

and

$$\|J_n(t, y) - J_n(t, y')\|_p \leq K_2 |y - y'|.$$

Proof. Firstly by Minkowski's integral inequality we have

$$\begin{aligned}\|J_n(t, y) - J_n(t', y')\|_p &\leq \frac{1}{n!} \int_{\mathbb{R}^n} \|g(z)\|_p \big|Q_t(y, z) - Q_{t'}(y', z)\big| \, \mathrm{d}z \\ &\leq \frac{K_{p,g}}{n!} \int_{\mathbb{R}^n} \big|Q_t(y, z) - Q_{t'}(y', z)\big| \, \mathrm{d}z.\end{aligned}$$

We first consider the time increment. Using the bound (3.36) on the time derivative of Q_t there is a constant C depending only on n such that for all $y \in \mathbb{R}^n$ and t, $t' \in [1/M, M]$

$$\begin{aligned}\int_{\mathbb{R}^n} \big|Q_t(y, z) - Q_{t'}(y, z)\big| \, \mathrm{d}z &= \int_{\mathbb{R}^n} \left| \int_t^{t'} \frac{\partial Q_r}{\partial r}(y, z) \, \mathrm{d}z \right| \mathrm{d}z \\ &\leq \int_{\mathbb{R}^n} \left| \int_t^{t'} \frac{C}{r} Q_{2r}(y, z) \, \mathrm{d}r \right| \mathrm{d}z \\ &\leq C \int_t^{t'} r^{-1} \int_{\mathbb{R}^n} Q_{2r}(y, z) \, \mathrm{d}z\mathrm{d}r \\ &\leq C n! M |t - t'|.\end{aligned}$$

Similarly for the space increment we need to estimate

$$\int_{\mathbb{R}^n} |Q_t(y, z) - Q_t(y', z)| \, \mathrm{d}z = \int_{\mathbb{R}^n} \left| \int_0^1 \nabla Q_t\big(r(\rho), z\big) \cdot r'(\rho) \, \mathrm{d}\rho \right| \mathrm{d}z,$$

where $r(\rho) = (1 - \rho)y + \rho y'$, $\rho \in [0, 1]$ is the straight line from y' to y. Equation (3.25) shows that for all $1 \leq i \leq n$

$$\frac{\partial Q_t}{\partial x_i}(y, z) = c_n (2\pi)^{-\frac{n}{2}} t^{-\frac{n^2+1}{2}} \Delta(z)^2 \int_{\mathcal{U}(n)} \frac{(U^\dagger D_z U)_{ii} - y_i}{\sqrt{t}} \exp\Big(-\frac{1}{2t} \mathrm{Tr}(D_z - U D_y U^\dagger)^2\Big) \, \mathrm{d}U.$$

The integrand above is bounded by

$$\begin{aligned}&2\frac{\big((U^\dagger D_z U)_{ii} - y_i\big)}{\sqrt{4t}} \prod_{j=1}^n \exp\Big(-\frac{1}{4t}\big((U^\dagger D_z U)_{jj} - y_j\big)^2\Big) \exp\Big(-\frac{1}{4t} \mathrm{Tr}(D_y - U^\dagger D_z U)^2\Big) \\ &\qquad \leq \sqrt{\frac{2}{e}} \exp\Big(-\frac{1}{4t} \mathrm{Tr}(D_y - U^\dagger D_z U)^2\Big).\end{aligned}$$

Therefore, using the Harish-Chandra formula again, we have for all i

$$\frac{\partial Q_t}{\partial y_i}(y,z) \le \sqrt{\frac{2}{e}}\frac{2^{n^2/2}}{\sqrt{t}} Q_{2t}(y,z).$$

Consequently, there is a constant $C > 0$ depending only on n such that for all $t \in [1/M, M]$ and y, $y' \in \mathbb{R}^n$

$$\begin{aligned}\int_{\mathbb{R}^n} |Q_t(y,z) - Q_t(y',z)| \,\mathrm{d}z &\le \frac{C}{\sqrt{t}} \int_0^1 |r'(\rho)| \int_{\mathbb{R}^n} Q_{2t}(r(\rho),z)\,\mathrm{d}z\,\mathrm{d}\rho \\ &\le Cn!\sqrt{M}|y-y'|,\end{aligned}$$

which completes the proof. □

We now turn our attention to the stochastic integral term $I_n(t,y)$.

Proposition 3.4.2. *Let $M > 1$ and $p \ge 2$. There exists a constant $K := K(g,M,n,p)$ such that for all (t,y) and $(u,z) \in [0,M] \times \mathbb{R}^n$*

$$\|I_n(t,y) - I_n(u,z)\|_p \le K\big(|t-u|^{1/4} + |y-z|^{1/2}\big).$$

Proof. We consider the spatial and temporal increment separately. By (3.14), there is a constant $C := C(g,M,n,p)$ such that

$$\sup_{(s,y')\in[0,M]\times\mathbb{R}^n} \|M_n(s,y')\|_p^2 \le C.$$

Then by Lemma 3.2.3 and Theorem 3.2.5(a)

$$\begin{aligned}\|I_n(t,y) - I_n(t,z)\|_p^2 &\le CA_n^2c_p^2 \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} Q_{t-s}(y,y') - Q_{t-s}(z,y')\,\mathrm{d}y'_* \right)^2 \mathrm{d}y'_1 \mathrm{d}s \\ &\le C_1CA_n^2c_p^2|y-z|.\end{aligned}$$

For the temporal increment we have two terms (assuming without loss of generality that $0 \le u \le t \le M$)

$$\|I_n(t,y) - I_n(u,y)\|_p^2 \le 2\mathrm{I} + 2\mathrm{II},$$

where by Theorem 3.2.5(b), there exists a C_2 such that

$$
\begin{aligned}
\mathrm{I} &:= \left\| A_n \int_0^u \int_{\mathbb{R}^n} \big(Q_{t-s}(y,y') - Q_{u-s}(y,y')\big) M_n(s,y')\, \mathrm{d}y'_*\, W(\mathrm{d}s, \mathrm{d}y'_1) \right\|_p^2 \\
&\leq C A_n^2 c_p^2 \int_0^u \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} Q_{t-s}(y,y') - Q_{u-s}(y,y')\, \mathrm{d}y'_* \right)^2 \mathrm{d}y'_1 \mathrm{d}s \\
&\leq C_2 C A_n^2 c_p^2 |t-u|^{1/2},
\end{aligned}
$$

and a constant C_3 such that

$$
\begin{aligned}
\mathrm{II} &:= \left\| A_n \int_u^t \int_{\mathbb{R}^n} Q_{t-s}(y,y') M_n(s,y')\, \mathrm{d}y'_*\, W(\mathrm{d}s, \mathrm{d}y'_1) \right\|_p^2 \\
&\leq C_3 C A_n^2 c_p^2 |t-u|^{1/2}.
\end{aligned}
$$

□

By the subadditivity of the function $x \mapsto |x|^\beta$, for $\beta \in (0,1]$ we have

$$
|y-y'|^\beta = \left(\sum_{i=1}^n |y_i - y'_i|^2 \right)^{\beta/2} \leq \sum_{i=1}^n |y_i - y'_i|^\beta.
$$

Lemma 3.4.1 and Proposition 3.4.2 together shows that for all $M > 1$ and $p \geq 2$, there is a constant $C := C(g, M, n, p)$ such that for all (t,y) and (t',y') in $[1/M, M] \times [-M, M]^n$,

$$
\|M_n(t,y) - M_n(t',y')\|_p \leq C \left(|t-t'|^{1/4} + \sum_{i=1}^n |y_i - y'_i|^{1/2} \right).
$$

Taking p large enough and applying Theorem 2.4.1 shows that M_n has a version that is locally Hölder continuous on $(0,\infty) \times \mathbb{R}^n$ with indices up to $1/4$ in time and up to $1/2$ in space.

3.4.2 Delta Initial Data

We now turn our attention to $M_n(t,x,y)$. Observe that in this case we cannot apply the method used in Proposition 3.4.2 directly since the pth moments of $M_n(t,x,y)$ are not bounded uniformly in time, for instance if $x = y$ then

$$
\|M_n(t,x,x)\|_2 \geq \frac{p_n^*(t,x,x)}{\Delta(x)^2} = \frac{(2\pi t)^{-n/2}}{\Delta(x)^2} \Big(1 + \sum_{\substack{\sigma \in S_n \\ \sigma \neq \mathrm{id}}} \mathrm{sgn}(\sigma) \prod_{i=1}^n e^{-(x_i - x_{\sigma(i)})^2/2t} \Big),
$$

which converges to infinity as $t \downarrow 0$. However, for any $t > 0$ fixed we have by (3.15) and Lemma 3.2.2 that there is a constant $C := C(n, p, t)$ such that

$$\|M_n(t,x,y)\|_p^2 \le 2\Big(\frac{p_n^*(t,x,y)}{\Delta(x)\Delta(y)}\Big)^2 \mathbb{E}_{x,y;t}^{X,Y}\Big[\exp\Big(2c_p^2 \sum_{i,j=1}^n L_t(X^i - Y^j)\Big)\Big] \le Ct^{-n^2},$$

uniformly for $x, y \in \mathbb{R}^n$. Thus, for all positive times, M_n belongs to the class of initial data in Theorem 3.1.2(a) and we can use the same method as in the previous chapter. Let $\tau > 0$ and consider the shifted white noise $\dot{W}^\tau(s, y) := \dot{W}(\tau+s, y)$. Define $M_n^\tau(t, x, y) := M_n(\tau+t, x, y)$ then it is easy to check in the same way as in Lemma 2.4.4 by using the semigroup property of Q_t that M_n^τ satisfies the integral equation

$$\begin{aligned} M_n^\tau(t,x,y) = &\frac{1}{n!}\int_{\mathbb{R}^n} M_n(\tau,x,y')Q_t(y,y')\,\mathrm{d}y' \\ &+ A_n \int_0^t \int_{\mathbb{R}^n} Q_{t-s}(y,y')M_n^\tau(s,x,y')\,\mathrm{d}y'_*\,W^\tau(\mathrm{d}s,\mathrm{d}y'_1). \end{aligned}$$

In other words, M_n^τ is the solution to (3.13) driven by the shifted noise $\dot{W}^\tau$ with initial condition $M_n^\tau(0, x, y) = M_n(\tau, x, y)$. Now define

$$\hat{\mathcal{M}}_n(t,x,y) := \begin{cases} M_n(t,x,y) & \text{if } 0 \le t \le \tau, \\ M_n^\tau(t-\tau,x,y) & \text{if } t > \tau. \end{cases}$$

Clearly, $\hat{\mathcal{M}}_n(t, x, y)$ solves (3.11) and by uniqueness, $\hat{\mathcal{M}}_n$ is a modification of the chaos series (3.9). Let $M > 1$ and $p \ge 2$ then since $\sup_{x,y\in\mathbb{R}^n} \|M_n(\tau, x, y)\|_p < \infty$, Lemma 3.4.1 and Proposition 3.4.2 applies to show that there is a constant $C := C(M, n, p, \tau)$ such that for all t, $t' \in [\tau, M]$ and y, $y' \in [-M, M]^n$ and $x \in \mathbb{R}^n$

$$\|M_n^\tau(t,x,y) - M_n^\tau(t',x,y')\|_p \le C\big(|t-t'|^{1/4} + |y-y'|^{1/2}\big). \tag{3.44}$$

Continuity in the Initial Condition

We study the continuity of $x \mapsto M_n(t, x, y)$; in fact we show that $(t, x, y) \mapsto M_n(t, x, y)$ is jointly continuous. Recall the chaos expansion of $M_n(t, x, y)$:

$$M_n(t,x,y) = J_n(t,x,y)\Big(1 + \sum_{k=1}^\infty \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s},\mathbf{y}';t,x,y)\,W^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{y}')\Big), \tag{3.45}$$

where for $0 < s_1 < \ldots < s_k < t$, $\mathbf{y} = (y_1^1, y_1^2, \ldots, y_1^k)$

$$\begin{aligned} &R_k(\mathbf{s},\mathbf{y};t,x,y) \\ &\quad = A_n^k \int_{(\mathbb{R}^{n-1})^k} \frac{p_n^*(s_1,x,y^1)\prod_{i=2}^k p_n^*(s_i-s_{i-1},y^{i-1},y^i)p_n^*(t-s_k,y^k,y)}{p_n^*(t,x,y)} \prod_{i=1}^k\prod_{j=2}^n \mathrm{d}y_j^i. \end{aligned}$$

It is easy to see that $J_n(t,x,y) = J_n(t,y,x)$ and from the expression of R_k, one can see that for all $k \geq 1$

$$R_k(\mathbf{s},\mathbf{y};t,x,y) = R_k(t-\tilde{\mathbf{s}},\tilde{\mathbf{y}};t,y,x), \tag{3.46}$$

where $t-\tilde{\mathbf{s}} := (t-s_k,\ldots,t-s_1)$, $0 < t-s_k < \ldots < t-s_1 < t$ and $\tilde{\mathbf{y}} := (y_1^k, y_1^{k-1},\ldots,y_1^1)$. Indeed, since $p_n^*(t,x,y) = p_n^*(t,y,x)$ for all (t,x,y), we have

$$\begin{aligned}
&R_k(\mathbf{s},\mathbf{y};t,x,y) \\
&\quad = A_n^k \int_{(\mathbb{R}^{n-1})^k} \frac{p_n^*(s_1,y^1,x)\prod_{i=2}^k p_n^*(s_i-s_{i-1},y^i,y^{i-1})p_n^*(t-s_k,y,y^k)}{p_n^*(t,y,x)} \prod_{i=1}^k\prod_{j=2}^n \mathrm{d}y_j^i \\
&\quad = R_k(t-\tilde{\mathbf{s}},\tilde{\mathbf{y}};t,x,y).
\end{aligned}$$

Therefore, it is reasonable to think that each term in the series (3.45) above is symmetric in x and y provided one can reverse time in the multiple stochastic integral. This motivates the following proposition.

Proposition 3.4.3. *For all $n \geq 1$ and $y \in \mathbb{R}^n$ the random fields $(M_n(t,x,y);(t,x) \in (0,\infty)\times\mathbb{R}^n)$ and $(M_n(t,y,x);(t,x) \in (0,\infty)\times\mathbb{R}^n)$ are equal in distribution.*

We first need the following intermediate step. Define a time reversed white noise $\tilde{W}$ by $\tilde{W}([0,s]\times A) = \dot{W}([t-s,t]\times A)$, $s \leq t$ and $A \in \mathscr{B}_b(\mathbb{R})$ then we have the following

Lemma 3.4.4. *Let $f \in L_S^2([0,t]^k \times \mathbb{R}^k)$ then*

$$\int_{[0,t]^k}\int_{\mathbb{R}^k} f(\mathbf{s},\mathbf{y})\, W^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{y}) = \int_{[0,t]^k}\int_{\mathbb{R}^k} f(t-\mathbf{s},\mathbf{y})\, \tilde{W}^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{y}) \quad \text{a.s.},$$

where $t-\mathbf{s} = (t-s_1,\ldots,t-s_k)$.

Proof. We first prove it in the case when f is an elementary function of the form

$$f(\mathbf{s},\mathbf{y}) := \sum_{\pi\in S_k}\prod_{i=1}^k 1\{(s_{\pi i},y_{\pi i}) \in [a_i,b_i]\times A_i\},$$

for disjoint sets $A_i \in \mathbb{R}$ and disjoint intervals $[a_i,b_i] \in [0,t]$, $i = 1,\ldots,k$. Then by definition, the integral of f is equal to (see Appendix A)

$$\begin{aligned}
(f\cdot W)_k(t) &= k!\prod_{i=1}^k \dot{W}([a_i,b_i]\times A_i) \\
&= k!\prod_{i=1}^k \tilde{W}([t-b_i,t-a_i]\times A_i) \\
&= \int_{[0,t]^k}\int_{\mathbb{R}^k}\sum_{\pi\in S_k}\prod_{i=1}^k 1\{(s_{\pi i},y_{\pi i}) \in [t-b_i,t-a_i]\times A_i\}\, \tilde{W}^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{y})
\end{aligned}$$

$$= \int_{[0,t]^k} \int_{\mathbb{R}^k} \sum_{\pi \in S_k} \prod_{i=1}^{k} 1\{(t - s_{\pi i}, y_{\pi i}) \in [a_i, b_i] \times A_i\} \, \tilde{W}^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y})$$

$$= \int_{[0,t]^k} \int_{\mathbb{R}^k} f(t - \mathbf{s}, \mathbf{y}) \, \tilde{W}^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}).$$

This proves the result for elementary functions. For general $f \in L^2_S([0,t]^k \times \mathbb{R}^k)$ we take a sequence $(f_n)_{n \geq 1}$ of elementary functions converging to f and use the convergence of $(f_n \cdot W)_k$ to $(f \cdot W)_k$. $\square$

Proof of Proposition 3.4.3. Fix $k \geq 1$ and $(t, x, y) \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$. Extend $R_k(\mathbf{s}, \mathbf{y}; t, x, y)$ to a function on $L^2([0,t]^k \times \mathbb{R}^k)$ by setting it to be zero for $\mathbf{s} \notin \Delta_k(t)$. Let $\tilde{R}_k$ be the symmetrisation of R_k given by

$$\tilde{R}_k(\mathbf{s}, \mathbf{y}; t, x, y) = \frac{1}{k!} \sum_{\pi \in S_k} R_k(\pi \mathbf{s}, \pi \mathbf{y}; t, x, y),$$

where $\pi \mathbf{s} = (s_{\pi(1)}, \ldots, s_{\pi(k)})$ and likewise for $\pi \mathbf{y}$. Clearly, we have $\tilde{R}_k(\tilde{\mathbf{s}}, \tilde{\mathbf{y}}; t, x, y) = \tilde{R}_k(\mathbf{s}, \mathbf{y}; t, x, y)$. Therefore by Lemma 3.4.4 and (3.46), (recall the definition of the multiple stochastic integral in Appendix A)

$$\begin{aligned}
\int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s}, \mathbf{y}; t, x, y) \, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) &= \int_{[0,t]^k} \int_{\mathbb{R}^k} \tilde{R}_k(\mathbf{s}, \mathbf{y}; t, x, y) \, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) \\
&= \int_{[0,t]^k} \int_{\mathbb{R}^k} \tilde{R}_k(t - \mathbf{s}, \mathbf{y}; t, x, y) \, \tilde{W}^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) \\
&= \int_{[0,t]^k} \int_{\mathbb{R}^k} \tilde{R}_k(\tilde{\mathbf{s}}, \tilde{\mathbf{y}}; t, y, x) \, \tilde{W}^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) \\
&= \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s}, \mathbf{y}; t, y, x) \, \tilde{W}^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}).
\end{aligned}$$

Thus, applying the above to each term of the sum in (3.45) we see that

$$\begin{aligned}
M_n(t, x, y) &= J_n(t, y, x) \left(1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s}, \mathbf{y}; t, y, x) \, \tilde{W}^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) \right) \\
&= M_n(t, y, x),
\end{aligned}$$

for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$ and the result follows. $\square$

Finally, we return to proving the joint continuity of the solution to (3.11). We bound $\|\hat{\mathcal{M}}_n(t, x, y) - \hat{\mathcal{M}}_n(t', x', y')\|_p^2$ by considering the increments in each variables separately. Since $\hat{\mathcal{M}}_n(t, x, y) = M_n^\tau(t - \tau, x, y)$ for $t \geq 2\tau$, we have by Proposition 3.4.3 and (3.44) that for all $M > 1$ and $p \geq 2$ there is a constant $C := C(M, n, p, \tau)$ such that for all (t, x, y) and

$(t', x', y') \in [2\tau, M] \times [-M, M]^n \times [-M, M]^n$

$$
\begin{aligned}
&\|\hat{\mathcal{M}}_n(t,x,y) - \hat{\mathcal{M}}_n(t',x',y')\|_p \\
&\quad \le \|M_n^\tau(t-\tau,x,y) - M_n^\tau(t'-\tau,x,y')\|_p + \|M_n^\tau(t'-\tau,y',x) - M_n^\tau(t'-\tau,y',x')\|_p \\
&\quad \le C\big(|t-t'|^{1/4} + |x-x'|^{1/2} + |y-y'|^{1/2}\big).
\end{aligned}
$$

Since $\tau > 0$ is arbitrary, we can take $2\tau = 1/M$ and thus we have shown that there exists a constant $\tilde{C} = \tilde{C}(M, n, p)$ such that for all (t, x, y) and $(t', x', y') \in [1/M, M] \times [-M, M]^{2n}$ the above inequality holds with $\tilde{C}$ in place of C. Finally, using the subadditivity of $x \mapsto |x|^\beta$ for $\beta \in (0, 1]$ and applying Theorem 2.4.1 proves the existence of a Hölder continuous version. This concludes the entire proof of Theorem 3.1.2.

Chapter 4

Strict Positivity

4.1 Introduction

In this chapter we prove the strict positivity part of Theorem 3.1.1. In the same way as for continuity we first prove the result for $M_n(t, x, y)$ and then use the fact that M_n and Z_n agree up to a multiplicative constant when all of the x and y coordinates coincide, see equation (3.10). In fact we are going to prove a strong comparison principle for solutions of the integral equation (3.13) or (3.11) from which the strict positivity result follows.

It is well known that the solution to the stochastic heat equation (SHE) is strictly positive which was first proved by Mueller in [Mue91]. More precisely, he proved that for each $t > 0$

$$\mathbb{P}[u(t, x) > 0 \text{ for every } x \in \mathbb{R}] = 1,$$

for solutions to the SHE with initial condition $f \geq 0$ being a continuous function with compact support and $f(x) > 0$ for some $x \in \mathbb{R}$. Shiga in [Shi94] proved the stronger statement

$$\mathbb{P}[u(t, x) > 0 \text{ for every } x \in \mathbb{R} \text{ and every } t > 0] = 1,$$

for initial data being continuous functions such that the tails grow no faster than $e^{\lambda|x|}$ for all $\lambda > 0$. More recently, Moreno Flores in [Flo14] proved the strict positivity of the solution for delta initial conditions, using a convergence result of a polymer model to the SHE, see [AKQ14b]. Chen and Kim [CK14] further generalised the strict positivity result to the fractional SHE for measure-valued initial data by adapting Shiga's method.

In all of the proofs above (except for the polymer proof) a key result is a large deviation estimate on the stochastic integral term of the solution. Mueller proved such result using the fact that integrals of the type $\int_0^t \int_{\mathbb{R}} f(s, y)\ W(\mathrm{d}s, \mathrm{d}y)$ can be considered as a time-changed Brownian motion. Chen and Kim using a method of [CJK12] derived a similar estimate for the fractional SHE using Kolmogorov's continuity criterion. We will follow the approach of [CK14] since we have already obtained the continuity estimates in the previous chapter.

The main result of this chapter is the following theorem.

Theorem 4.1.1. *Let g be as in Theorem 3.1.2(a) with the additional property that g is non-negative almost surely and $\mathbb{P}[g(y) > 0$ for some $y \in \mathbb{R}^n] = 1$. Then the solution M_n^g to (3.13) satisfies*

$$\mathbb{P}[M_n^g(t, y) > 0 \text{ for all } t > 0 \text{ and } y \in \mathbb{R}^n] = 1.$$

Let M_n be the random field defined by the chaos series (3.9) then

$$\mathbb{P}[M_n(t, x, y) > 0 \text{ for all } t > 0 \text{ and } x, y \in \mathbb{R}^n] = 1.$$

The strict positivity of Z_n now follows from (3.10) and an immediate consequence of this and the continuity result of the previous chapter is

Corollary 4.1.2. *For all $n \geq 1$, $h_n(t, x) := \log\big(Z_n(t, 0, x)/Z_{n-1}(t, 0, x)\big)$ with $Z_0 = 1$ is well defined and it is a continuous function of (t, x) over $(0, \infty) \times \mathbb{R}$.*

The outline of the chapter is as follows. In the next section we prove a weak comparision principle for the integral equation (3.13) which will be used repeatedly, then in Section 4.3 we prove a strong comparison principle of which the strict positivity of M_n is a corollary.

4.2 A Weak Comparision Principle

Recall that $K_n(t, x, y)$ can be expressed as $K_n(t, x, y) = \det[u(t, x_i, y_j)]_{i,j=1}^n$ where $u(t, x, y)$ is the solution to (3.2) with initial data δ_x. Bertini–Cancrini [BC95] proved that $u(t, x, y)$ is the limit in $L^p(\Omega)$ for all $p \geq 2$ of $u^\varepsilon(t, x, y)$ as $\varepsilon \to \infty$, where $u^\varepsilon(t, x, y)$ is the solution to the stochastic heat equation subject to a mollified white noise W^ε in place of the space-time white noise. Its solution is given by the following Feymann–Kac formula which is well defined for the noise W^ε:

$$u^\varepsilon(t, x, y) = p_t(x - y)\mathbb{E}^b_{x,y;t}\left[\mathcal{E}\text{xp}\Big(\int_0^t W^\varepsilon(s, b_s)\, \mathrm{d}s\Big)\right],$$

where the expectation is with respect to a Brownian bridge b starting from x at time 0 and ending in y at time t. By the above Feymann–Kac formula it is then clear that for all $(t, x, y) \in (0, \infty) \times \mathbb{R} \times \mathbb{R}$, with probability 1, $u(t, x, y) \geq 0$. Using this and the determinant formula for K_n, the authors in [OW11, Proposition 5.5] proved by a path switching argument that $K_n(t, x, y) \geq 0$ almost surely, for all $(t, x, y) \in (0, \infty) \times W_n \times W_n$.

In fact, a stronger result is true since the above implies that $K_n(t, x, y) \geq 0$ for all rational points (t, x, y) almost surely. It is well known that $(t, x, y) \mapsto u(t, x, y)$ has a jointly continuous version and hence the same is true for K_n as it is just a sum of products of the

u's. Therefore, by continuity

$$\mathbb{P}[K_n(t,x,y) \geq 0 \text{ for all } t > 0 \text{ and } x, y \in W_n] = 1.$$

Since the Vandermonde determinant is non-negative on W_n, we see that the same is true for M_n in the interior W_n°. By the continuity of M_n proved in the previous section, this non-negativity extends to the boundary of the Weyl chamber and by symmetry to the whole of $\mathbb{R}^n$. That is,

$$\mathbb{P}[M_n(t,x,y) \geq 0 \text{ for all } t > 0 \text{ and } x, y \in \mathbb{R}^n] = 1. \tag{4.1}$$

The next lemma extends this to solutions $M_n^g(t,y)$ of equation (3.13) with non-negative initial data g and in fact by the linearity of the equation this is equivalent to a weak comparison principle.

Lemma 4.2.1 (Weak comparison principle)**.** *Let $M_n^1(t,y)$ and $M_n^2(t,y)$ be the solution to (3.13) with symmetric initial data g_1 and g_2 respectively. If $g_1 \geq g_2$, then*

$$\mathbb{P}[M_n^1(t,y) \geq M_n^2(t,y) \text{ for all } t > 0 \text{ and } y \in \mathbb{R}^n] = 1.$$

Proof. By linearity of the equation (3.13), it suffices to prove the lemma in the case $g \geq 0$. For $(t,y) \in [0,\infty) \times \mathbb{R}^n$, define

$$v_g(t,y) := \frac{1}{n!} \int_{\mathbb{R}^n} g(x) M_n(t,x,y) \Delta(x)^2 \, \mathrm{d}x.$$

A direct calculation shows that v_g satisfies the mild equation (3.13) and so by uniqueness $v_g(t,y) = M_n^g(t,y)$ almost surely for all $(t,y) \in [0,\infty) \times \mathbb{R}^n$. Now by (4.1) and the non-negativity of g, it is clear that for all $(t,y) \in [0,\infty) \times \mathbb{R}^n$, $v_g(t,y) \geq 0$ almost surely. This and the continuity of $(t,y) \mapsto M_n^g(t,y)$ shows that $\mathbb{P}[M_n^g(t,y) \geq 0 \text{ for all } t \geq 0 \text{ and } y \in \mathbb{R}^n] = 1$ as required. □

4.3 A Strong Comparison Principle

We now prove a strong comparision principle of which Theorem 4.1.1 is an easy corollary.

Theorem 4.3.1 (Strong comparision principle)**.**

(a) Let $M_n^1(t,y)$ and $M_n^2(t,y)$ be two solutions to (3.13) with initial data g_1 and g_2 respectively where g_1 and g_2 are as in Theorem 3.1.2(a). If furthermore $g_1 \geq g_2$ and $g_1(y) > g_2(y)$ for some $y \in \mathbb{R}^n$ almost surely, then

$$\mathbb{P}[M_n^1(t,y) > M_n^2(t,y) \text{ for all } t > 0 \text{ and } y \in \mathbb{R}^n] = 1.$$

(b) Let $M_n(t,x,y)$ be the solution to (3.11), then

$$\mathbb{P}[M_n(t,x,y) > 0 \text{ for all } t > 0 \text{ and } x, y \in \mathbb{R}^n] = 1.$$

We begin with a lemma which provides a lower bound for the deterministic term $J_n(t,y)$ in (3.13).

Lemma 4.3.2. *Let $\beta := \beta(n) = \mathbb{P}_{\mathrm{GUE}}[\phi_i(Y) \geq 0 \text{ for all } i]/2 > 0$ where $\phi_i(Y)$ is the ith eigenvalue of an $n \times n$ matrix Y from the Gaussian Unitary Ensemble (GUE). For all $h > 0$, $t > 0$, $M > 0$, there exists an $m_0 := m_0(h, M, n, t)$ such that for all $m \geq m_0$, all $s \in [t/2m, t/m]$ and $x \in W_n$,*

$$\int_{W_n} Q_s(x,y) 1_{(-h,h)^n}(y)\, \mathrm{d}y \geq \beta 1_{(-h-M/m, h+M/m)^n}(x).$$

Proof. Since Dyson Brownian motion is realised as the eigenvalues of Brownian motion on the space of $n \times n$ Hermitian matrices $\mathcal{H}(n)$, we have that

$$\int_{W_n} Q_s(x,y) 1_{(-h,h)^n}(y)\, \mathrm{d}y = \int_{\mathcal{H}(n)} P_s(Y) 1_{(-h,h)^n}\big(\phi(Y + D_x)\big)\, \mathrm{d}Y,$$

where $P_s(A - B) = 2^{-n/2}(\pi s)^{-n^2/2} e^{-\mathrm{Tr}(A-B)^2/2s}$ for $A, B \in \mathcal{H}(n)$ is the transition density of Brownian motion on the space of Hermitian matrices and $\phi : \mathcal{H}(n) \to W_n$ is such that $\phi(Y) = y = (y_1, \ldots, y_n) = (\phi_1(Y), \ldots, \phi_n(Y))$ is the vector of ordered eigenvalues of Y. D_x is the diagonal matrix with entries $x = (x_1, \ldots, x_n)$. Weyl's eigenvalue inequality [Bha97, Theorem III.2.1] implies that for two Hermitian matrices A, B with eigenvalues $\phi_i(A)$ and $\phi_i(B)$, $1 \leq i \leq n$ respectively, the following hold

$$\phi_1(A+B) \leq \phi_1(A) + \phi_1(B) \quad \text{and} \quad \phi_n(A) + \phi_n(B) \leq \phi_n(A+B).$$

Therefore

$$\begin{aligned} 1_{(-h,h)^n}\big(\phi(Y + D_x)\big) &= 1\big\{\phi_n(Y + D_x) \geq -h\big\} 1\big\{\phi_1(Y + D_x) \leq h\big\} \\ &\geq 1\big\{\phi_n(Y) + x_n \geq -h\big\} 1\big\{\phi_1(Y) + x_1 \leq h\big\}, \end{aligned}$$

and hence

$$\int_{W_n} Q_s(x,y)1_{(-h,h)^n}(y)\,\mathrm{d}y$$
$$\geq \int_{\mathcal{H}(n)} P_s(Y)\,1\big\{\phi_n(Y) \geq -h - x_n\big\}1\big\{\phi_1(Y) \leq h - x_1\big\}\,\mathrm{d}Y$$
$$= \int_{\mathcal{H}(n)} P_1(Y)\,1\Big\{\phi_n(Y) \geq \frac{-h-x_n}{\sqrt{s}}\Big\}1\Big\{\phi_1(Y) \leq \frac{h-x_1}{\sqrt{s}}\Big\}\,\mathrm{d}Y$$
$$= \int_{\mathcal{H}(n)} P_1(Y)\,\prod_{i=1}^{n} 1\Big\{\phi_i(Y) \in \Big(\frac{-h-x_n}{\sqrt{s}}, \frac{h-x_1}{\sqrt{s}}\Big)\Big\}\,\mathrm{d}Y.$$

Let $\beta > 0$ be the constant in the statement of the lemma then for $-h - M/m \leq x_i \leq 0$, $1 \leq i \leq n$ and $t/2m \leq s \leq t/m$, we have

$$\int_{W_n} Q_s(x,y)1_{(-h,h)^n}(y)\,\mathrm{d}y$$
$$\geq \int_{\mathcal{H}(n)} P_1(Y)\prod_{i=1}^{n} 1\{\phi_i(Y) \in (\sqrt{2}M(tm)^{-1/2}, h(m/t)^{1/2})\}\,\mathrm{d}Y. \tag{4.2}$$

Similarly, for $0 \leq x_i \leq h + M/m$, $1 \leq i \leq n$ and s in the same range as above, we have

$$\int_{W_n} Q_s(x,y)1_{(-h,h)^n}(y)\,\mathrm{d}y$$
$$\geq \int_{\mathcal{H}(n)} P_1(Y)\prod_{i=1}^{n} 1\{\phi_i(Y) \in (-h(m/t)^{1/2}, -\sqrt{2}M(tm)^{-1/2})\}\,\mathrm{d}Y. \tag{4.3}$$

Taking m large enough and noting that $P_1(Y)$ is the probability density of a GUE matrix Y, we see that both (4.2) and (4.3) can be made greater than β and hence completes the proof. □

Lemma 4.3.3. *Let β be the constant in Lemma 4.3.2. Let $t > 0$, $M > 0$ and $h > 0$ be such that $(-h,h) \subseteq (-2M, 2M)$ and let M_n be the solution to (3.13) with initial data $g = 1_{(-h,h)^n}$. Then, there exists an $m_0 := m_0(h, M, n, t)$ such that for all $m \geq m_0$*

$$\mathbb{P}\Big[M_n(s,y) \geq \frac{\beta}{2}1_{(-h-M/m,h+M/m)^n}(y) \text{ for all } t/2m \leq s \leq t/m \text{ and } y \in \mathbb{R}^n\Big] \geq 1 - \delta(m),$$

where $\delta(m)$ is such that $(1-\delta(m))^m \to 1$ as $m \to \infty$.

Proof. Let β be as in Lemma 4.3.2 and let $M > 0$, $t > 0$, $h > 0$ be given, then by Lemma 4.3.2 there exist an $m_0 = m_0(h, M, n, t)$ such that for all $m \geq m_0$, all $s \in [t/2m, t/m]$ and $y \in \mathbb{R}^n$

$$J_n(s,y) \geq \beta 1_{(-h-M/m,h+M/m)^n}(y).$$

Since J_n is deterministic, we have

$$\begin{aligned}
&\mathbb{P}\Big[M_n(s,y) < \frac{\beta}{2}1_{(-h-M/m,h+M/m)^n}(y) \text{ for some } s \in [t/2m, t/m] \text{ and } y \in \mathbb{R}^n\Big] \\
&\quad \le \mathbb{P}\Big[I_n(s,y) < -\frac{\beta}{2}1_{(-h-M/m,h+M/m)^n}(y) \text{ for some } s \in [t/2m, t/m] \text{ and } y \in \mathbb{R}^n\Big] \\
&\quad \le \mathbb{P}\left[\sup_{\substack{s\in[t/2m,t/m]\\ y\in(-h-M/m,h+M/m)^n}} |I_n(s,y)| > \frac{\beta}{2}\right] \\
&\quad \le \left(\frac{\beta}{2}\right)^{-p} \mathbb{E}\left[\sup_{\substack{s\in[t/2m,t/m]\\ y\in(-h-M/m,h+M/m)^n}} |I_n(s,y)|^p\right] \\
&\quad \le \left(\frac{\beta}{2}\right)^{-p} \mathbb{E}\left[\sup_{(s,y)\in[t/2m,t/m]\times[-3M,3M]^n} |I_n(s,y)|^p\right], \qquad (4.4)
\end{aligned}$$

for all $p \ge 2$ by Chebychev's inequality. We shall bound the final expectation. Fix $\theta \in \left(0, \frac{1}{4} - \frac{n+1}{p}\right)$ then since $I_n(0,y) \equiv 0$ for all y, we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{\substack{s\in[t/2m,t/m]\\ y\in[-3M,3M]^n}} \left|\frac{I_n(s,y)}{(t/m)^\theta}\right|^p\right] &\le \mathbb{E}\left[\sup_{\substack{s\in[t/2m,t/m]\\ y\in[-3M,3M]^n}} \left|\frac{I_n(s,y) - I_n(0,y)}{s^\theta}\right|^p\right] \\
&\le \mathbb{E}\left[\sup_{\substack{s,s'\in[0,t/m],(s,y)\neq(s',y')\\ y,y'\in[-3M,3M]^n}} \left|\frac{I_n(s,y) - I_n(s',y')}{|s-s'|^\theta + |y-y'|^\theta}\right|^p\right]. \qquad (4.5)
\end{aligned}$$

Recall that Kolmogorov's continuity criterion (see [RY99, Theorem 2.1]) states that for a stochastic process $(X(t) : t \in [0,T]^d)$, if there exist strictly positive constants C, α and p with $\alpha p > d$ such that

$$\|X(s) - X(t)\|_p \le C|s-t|^\alpha, \quad \text{for all } s,t \in [0,T]^d,$$

then X has a Hölder continuous modification which satisfies for all $\theta \in [0, \alpha - d/p)$,

$$\left\|\sup_{\substack{s\neq t\\ s,t\in[0,T]^d}} \frac{|X(s)-X(t)|}{|s-t|^\theta}\right\|_p \le CT^{\alpha-\theta} \frac{2^{\theta+1}2^{d/p}}{1 - 2^{d/p}2^{-(\alpha-\theta)}}. \qquad (4.6)$$

Note that for θ fixed, the right hand side of (4.6) is bounded for all $p \ge 2$.

From the proof of Proposition 3.4.2 we see that for all $p \ge 2$ there is a constant $C := C(n)$ such that for all (s,y), $(s',y') \in [0,t/m] \times [-3M,3M]^n$,

$$\|I_n(s,y) - I_n(s',y')\|_p \le Cc_p \sup_{\substack{s\in[0,t/m]\\ y\in[-3M,3M]^n}} \|M_n(s,y)\|_p \big(|s-s'|^{1/4} + |y-y'|^{1/2}\big). \qquad (4.7)$$

Then by Kolmogorov's continuity criterion, for $p > 4(n+1)$ there is a constant $K' := K'(M,m,n,t)$ such that (4.5) is bounded by

$$(K')^p c_p^p \sup_{\substack{s\in[0,t/m]\\ y\in[-3M,3M]^n}} \|M_n(s,y)\|_p^p \leq (4K'\sqrt{p})^p e^{Ap^3 t/m},$$

for a constant A depending only on n, where to obtain the inequality we have used the moment bound (3.14) and the fact that $g \leq 1$, $|\mathrm{erf}(\cdot)| \leq 1$ and $c_p \leq 2\sqrt{p}$. Furthermore, if $m > m_0 \wedge t$ then $t/m \leq 1$ and thus for such m we can, by the explicit bound on the right hand side (4.6), replace the constant K' in the previous display with a constant $K := K(M,n)$. Consequently, for all $p > 4(n+1)$

$$\begin{aligned}\left(\frac{\beta}{2}\right)^{-p} \mathbb{E}\left[\sup_{\substack{s\in[0,t/m]\\ y\in[-3M,3M]}} |I_n(s,y)|^p\right] &\leq \left(\frac{8K\sqrt{p}}{\beta}\left(\frac{t}{m}\right)^{\theta}\right)^p e^{Ap^3 t/m}\\ &\leq \exp\left(\frac{Ap^3 t}{m} + p\log(8K\beta^{-1}t^{\theta}\sqrt{p}) - p\theta\log(m)\right).\end{aligned}$$

Choose $p = 8(n+1)$ and $\theta \in (\frac{1}{p}, \frac{1}{8})$ and for such choice denote the exponential in the last line above by $\delta(m)$, then for m large, $\delta(m) \approx \exp(-\log(m^{\rho(n+1)}))$ with $\rho = 8\theta > 1/(n+1)$ and therefore

$$(1-\delta(m))^m \approx \left(1 - \frac{1}{m^{\rho(n+1)}}\right)^m \to 1, \quad \text{as } m \to \infty,$$

for all $n \geq 1$ as required. □

We are now ready to prove the main result of this section.

Proof of Theorem 4.3.1. By linearity, $M_n^1 - M_n^2$ is the solution to (3.13) with initial data $g_1 - g_2$ and so it suffices to prove that $\mathbb{P}[M_n^g(t,y) > 0$ for all $t > 0$ and $y \in \mathbb{R}^n] = 1$, for g such that $g \geq 0$ and $g(y) > 0$ for some $y \in \mathbb{R}^n$ almost surely.

We first consider the case when g is a continuous function such that $g \geq 0$ and $g(y) > 0$ for some $y \in \mathbb{R}^n$ so that one can find constants $c > 0$, $d > 0$ small enough such that $g(x) \geq c\prod_{i=1}^n 1_{(y_i-d,y_i+d)}(x)$ for all $x \in \mathbb{R}^n$. Without loss of generality, we can assume $c = 1$ and take y to be the origin for convenience. By the weak comparision principle (Lemma 4.2.1), we can assume that the initial data is $g(\cdot) = 1_{(-d,d)^n}(\cdot)$. From now on we drop the superscript g and just write $M_n(t,y)$.

Let $\gamma = \beta/2$ where β is the constant in Lemma 4.3.2. Fix $t > 0$ and $M > 0$ such that $(-d,d) \subset (-M,M)$. For $k = 1,\ldots,m$, define the events

$$A_k := \left\{M_n(s,y) \geq \gamma^k 1_{(-d-\frac{Mk}{m},d+\frac{Mk}{m})^n}(y) \text{ for all } s \in \left[\frac{(2k-1)t}{2m}, \frac{kt}{m}\right] \text{ and } y \in \mathbb{R}^n\right\},$$

and for $k = 2, \dots, m$ the events

$$B_1 := \left\{ M_n(t/2m, y) \geq \gamma 1_{(-d-\frac{M}{m}, d+\frac{M}{m})^n}(y) \text{ for all } y \in \mathbb{R}^n \right\}$$

$$B_k :=$$

$$\left\{ M_n(s, y) \geq \gamma^k 1_{(-d-\frac{Mk}{m}, d+\frac{Mk}{m})^n}(y) \text{ for all } s \in \left[\frac{(k-1)t}{m}, \frac{(2k-1)t}{2m} \right] \text{ and } y \in \mathbb{R}^n \right\}.$$

We consider first the sets A_k. By Lemma 4.3.3, there is an m_0 such that for all $m \geq m_0$ there is a $\delta(m)$ such that

$$\mathbb{P}[A_1] \geq 1 - \delta(m).$$

Now assume that $A_1 \cap \cdots \cap A_{k-1}$ occurs. On the event A_{k-1} we have $M_n((k-1)t/m, y) \geq \gamma^{k-1} 1_{(-d-M(k-1)/m, d+M(k-1)/m)^n}(y)$ for all $y \in \mathbb{R}^n$ almost surely. Define a time shifted white noise by $\dot{W}^k(s, y) = \dot{W}((k-1)t/m + s, y)$. Let $M_n^k(s, y)$ be the solution driven by the noise $\dot{W}^k$ with initial data given by $\gamma^{k-1} 1_{(-d-M(k-1)/m, d+M(k-1)/m)^n}(y)$. On the event A_{k-1}, by the weak comparision principle, $M_n((k-1)t/m+s, y) \geq M_n^k(s, y)$ for all $s \geq 0$ and $y \in \mathbb{R}^n$ almost surely. It is easy to see that $\tilde{M}_n^k(s, y) := \gamma^{-(k-1)} M_n^k(s, y)$ is the solution to (3.13) with initial data $1_{(-d-M(k-1)/m, d+M(k-1)/m)^n}(y)$. Lemma 4.3.3 applied to $\tilde{M}_n^k$ with $h = d + M(k-1)/m$ shows that with the same m_0 and $\delta(\cdot)$ as above that for all $m \geq m_0$

$$\mathbb{P}\left[\tilde{M}_n^k(s, y) \geq \gamma 1_{(-d-\frac{Mk}{m}, d+\frac{Mk}{m})^n}(y) \text{ for all } s \in \left[\frac{t}{2m}, \frac{t}{m} \right] \text{ and } y \in \mathbb{R}^n \right] \geq 1 - \delta(m),$$

and hence

$$\mathbb{P}\left[M_n^k(s, y) \geq \gamma^k 1_{(-d-\frac{Mk}{m}, d+\frac{Mk}{m})^n}(y) \text{ for all } s \in \left[\frac{t}{2m}, \frac{t}{m} \right] \text{ and } y \in \mathbb{R}^n \right] \geq 1 - \delta(m).$$

By the above discussion, this implies that

$$\mathbb{P}[A_k | A_1 \cap \cdots \cap A_{k-1}] \geq 1 - \delta(m) \quad \text{for } 1 \leq k \leq m.$$

Now since the event A_1 implies the event B_1, $\mathbb{P}[B_1] \geq 1 - \delta(m)$ and then proceeding in the same manner as before, we have

$$\mathbb{P}[B_k | B_1 \cap \cdots \cap B_{k-1}] \geq 1 - \delta(m) \quad \text{for } 1 \leq k \leq m.$$

Finally, by the union bound

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{k=1}^m A_k \cap \bigcap_{k=1}^m B_k \right] &= 1 - \mathbb{P}\left[\left(\bigcap_{k=1}^m A_k \right)^c \cup \left(\bigcap_{k=1}^m B_k \right)^c \right] \\
&\geq 1 - \left(1 - \mathbb{P}\left[\bigcap_{k=1}^m A_k \right] \right) - \left(1 - \mathbb{P}\left[\bigcap_{k=1}^m B_k \right] \right) \\
&\geq 2\big(1 - \delta(m)\big)^m - 1.
\end{aligned}$$

Since $(1-\delta(m))^m \to 1$ as $m \to \infty$, we conclude that

$$\mathbb{P}\big[M_n(s,y) > 0 \text{ for all } s \in (0,t] \text{ and } y \in [-M,M]^n\big] \geq \lim_{m\to\infty} \mathbb{P}\left[\bigcap_{k=1}^m A_k \cap \bigcap_{k=1}^m B_k\right] = 1.$$

Since $t > 0$ and $M > 0$ are arbitrary, this completes the proof in the case when the initial data g is a continuous function.

We now prove the result for g satisfying the assumptions in Theorem 4.3.1(a). The idea is that after a small time $\tau > 0$, we are back in the situation above. We shall prove that for all $\tau > 0$,

$$\mathbb{P}[M_n(t,y) > 0 \text{ for all } t > \tau \text{ and } y \in \mathbb{R}^n] = 1. \tag{4.8}$$

and since τ is arbitrary this would imply the desired result. Let $\dot{W}^\tau(s,y) = \dot{W}(\tau+s,y)$ be the time shifted white noise and let M_n^τ be the solution to (3.13) driven by the noise $\dot{W}^\tau$ and with initial data $M_n(\tau,\cdot)$. The weak comparison principle shows that $\mathbb{P}[M_n(t,y) \geq 0$ for all $t \geq 0$ and $y \in \mathbb{R}^n] = 1$. We claim that $\mathbb{P}[M_n(\tau,y) > 0$ for some $y] = 1$ then since $y \mapsto M_n(\tau,y)$ is continuous, the strong comparison principle for continuous initial data proved above applied to the solution M_n^τ shows that $\mathbb{P}[M_n^\tau(s,y) > 0$ for all $s > 0$ all $y \in \mathbb{R}^n] = 1$ which proves (4.8).

Therefore, it remains to prove the claim. Suppose the opposite is true, that is $\mathbb{P}[M_n(\tau,y) = 0$ for all $y] > 0$ and consider the solution $M_n(s,\cdot)$ at time $s \leq \tau$. If $M_n(s,y) > 0$ for some y almost surely then the strong comparison principle for continuous initial data applies to show that $M_n(\tau,y) > 0$ for all y almost surely. Hence, $\mathbb{P}[M_n(s,y) = 0$ for all $y] > 0$ for all $0 \leq s \leq \tau$ which implies that $M_n(0,\cdot) \equiv 0$ with strictly positive probability which is a contradiction. Thus, we must have that $\mathbb{P}[M_n(\tau,y) = 0$ for all $y] = 0$ which proves the claim.

We now prove part (b) of the theorem; the everywhere strict positivity of $M_n(t,x,y)$. Fix $\tau > 0$ then the same argument as above together with Proposition 3.4.3 shows that there is a set N of probability zero such that on its complement, M_n is jointly continuous and $M_n(\tau,x,0) > 0$ for all x. Define $c(x) := M_n(\tau,x,0)/2$ and $d(x) = \inf\{|y| : y \in \mathbb{R}^n \text{ with } M_n(\tau,x,y) = c(x)\}$, then on the complement of N, $c(x)$ and $d(x)$ are strictly positive and $M_n(\tau,x,y) \geq c 1_{(-d,d)^n}(y)$ for all x, $y \in \mathbb{R}^n$. For $N \geq 1$, define the random set $B_N := \{x \in \mathbb{R}^n : c(x) \geq 1/N, d(x) \geq 1/N\}$ then $M_n(\tau,x,y) \geq (1/N)1_{(-1/N,1/N)^n}(y)$ for all y and all $x \in B_N$ almost surely. The strict positivity result proved above applied to the solution with initial data $(1/N)1_{(-1/N,1/N)^n}(y)$ together with the weak comparision principle implies that

$$\mathbb{P}[E_N] := \mathbb{P}[M_n(\tau+s,x,y) > 0 \text{ for all } s > 0 \text{ and } y \in \mathbb{R}^n, x \in B_N] = 1.$$

On the set N^c we have $\bigcup_{N=1}^\infty B_N = \mathbb{R}^n$ otherwise there exists an $x \in \mathbb{R}^n$ such that either $c(x) = 0$ or $d(x) = 0$ which is a contradiction and therefore $\mathbb{P}[\bigcap_{N=1}^\infty E_N] = \mathbb{P}[M_n(\tau + s,x,y) > 0$ for all $s > 0$ and $x,y \in \mathbb{R}^n] = 1$ as required. $\square$

Chapter 5

Markov Property

5.1 Introduction

Recall the process $Z_n(t,x,y)$ defined for $n = 1, 2, \ldots$, $t > 0$, x, $y \in \mathbb{R}$ by

$$Z_n(t,x,y) = p_t(x-y)^n \left(1 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k(\mathbf{s},\mathbf{y};t,x,y)\, W^{\otimes k}(\mathrm{d}\mathbf{s},\mathrm{d}\mathbf{y})\right)$$

where R_k is the k-point correlation function for a collection of n non-intersecting Brownian bridges all starting from x at time 0 and ending at y at time t. The results of the previous chapters have shown that Z_n is everywhere strictly positive and continuous over $(0,\infty) \times \mathbb{R} \times \mathbb{R}$ and that $M_n(t, x\mathbf{1}, y\mathbf{1}) = c_{n,t} Z_n(t,x,y)$ where $c_{n,t} = c_n t^{-n(n-1)/2}$, $c_n^{-1} = \prod_{i=1}^{n-1} i!$. It was shown in [OW11, Proposition 3.3 and 3.7] by considering a smooth space-time potential that $(Z_n, n \geq 1)$ should satisfy a system of coupled SPDEs, however unfortunately it is not immediately obvious that such SPDEs make sense in the white noise setting. Nevertheless, it does suggests that the process should have a Markovian evolution. Indeed, we have the following theorem which is the main result of this chapter.

Theorem 5.1.1. *For each $n \geq 1$ and $x \in \mathbb{R}$,*

$$\big(Z_1(t,x,\cdot), \ldots, Z_n(t,x,\cdot); t \geq 0\big), \tag{5.1}$$

is a Markov process with respect to the filtration generated by the space-time white noise with state space $\mathcal{C} := C(\mathbb{R}) \times \cdots \times C(\mathbb{R})$, where $C(\mathbb{R}) := C\big(\mathbb{R}, (0,\infty)\big)$ is the space of continuous functions from $\mathbb{R}$ to $(0,\infty)$.

In the case of the stochastic heat equation ($n = 1$), the Markov property can be seen from the Feynman–Kac formula since $u(s+t,x,y)$ can be written in the form

$$\mathbb{E}^X_{x,y;s+t}\Big[\mathscr{E}\mathrm{xp}\big(F_X(0,s)\big)\mathscr{E}\mathrm{xp}\big(F_X(s,s+t)\big)\Big],$$

where $F_X(s, s+t)$ is a function of the Brownian bridge X starting from X_s and ending at y and the white noise over the time interval $[s, s+t]$ which is independent of the white noise over $[0, s]$ and the bridge from x to X_s. However, this argument does not apply for $n \geq 2$ since the definition of Z_n involves non-intersecting Brownian bridges with the same starting and ending points but at any intermediate time each of the bridges are at distinct locations. Nevertheless, Theorem 5.1.1 is true and we shall prove it by deriving a formula which relates the process (5.1) to M_n which does satisfy a rigorous evolution equation and from which the Markov property follows naturally.

Throughout this chapter we denote vectors by boldface letters, for example $\mathbf{x}$ will always mean a vector $(x_1, \dots, x_d) \in \mathbb{R}^d$ for some d. For $\mathbf{z} \in W_{n-1}$ and $\mathbf{y} \in W_n$, write $\mathbf{z} \prec \mathbf{y}$ if $y_1 \geq z_1 > y_2 \geq \dots > y_{n-1} \geq z_{n-1} > y_n$. For $\mathbf{y} \in W_n^\circ$, denote by $\mathrm{GT}(\mathbf{y})$ the Gelfand–Tsetlin polytope

$$\mathrm{GT}(\mathbf{y}) := \{(\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^{n-1}) \in W_1 \times W_2 \times \dots \times W_{n-1} : \mathbf{y}^1 \prec \mathbf{y}^2 \prec \dots \prec \mathbf{y}^{n-1} \prec \mathbf{y}\}.$$

The key to the proof of the Markov property is the following integral formula.

Theorem 5.1.2. *The following holds for all $n \geq 1$, $t > 0$, $x \in \mathbb{R}$ and $\mathbf{y} \in W_n^\circ$*

$$M_n(t, x\mathbf{1}, \mathbf{y}) = \frac{1}{\Delta(\mathbf{y})} \prod_{i=1}^n u(t, x, y_i) \int_{\mathrm{GT}(\mathbf{y})} \prod_{k=1}^{n-1} \prod_{i=1}^{n-k} \frac{1}{t} \frac{Z_{k-1}(t, x, z_i^{n-k}) Z_{k+1}(t, x, z_i^{n-k})}{Z_k(t, x, z_i^{n-k})^2} \, \mathrm{d}z_i^{n-k}. \tag{5.2}$$

The integral formula (5.2) has a limit as $\mathbf{y}$ tends to the boundary ∂W_n of the Weyl chamber. In particular, as $\mathbf{y} \to y\mathbf{1}$, the right hand side converges to $c_{n,t} Z_n(t, x, y)$ which agrees with (3.10). This is a consequence of the fact that the volume of $\mathrm{GT}(\mathbf{y})$ is equal to $c_n \Delta(\mathbf{y})$, the continuity of Z_n and Lebesgue's differentiation theorem [Fol84, Theorem 3.21]. Thus, the integral formula recovers the value of $M_n(t, x\mathbf{1}, \cdot)$ on the whole of the Weyl chamber from the boundary values $\big(Z_1(t, x, \cdot), \dots, Z_n(t, x, \cdot)\big)$.

Theorem 5.1.2 was conjectured and proved only in the case $n = 2$ in [OW11], the obstacle being that the continuity of M_n on the whole of the Weyl chamber was only established in the $L^2(\Omega)$ sense and a proof of its strict positivity was unavailable. This was resolved in the previous chapters and here we are able to prove Theorem 5.1.2 for general $n \geq 2$. The proof takes as input the continuity and strict positivity of M_n and the determinantal expression

$$M_n(t, \mathbf{x}, \mathbf{y}) = \frac{\det[u(t, x_i, y_j)]_{i,j=1}^n}{\Delta(\mathbf{x}) \Delta(\mathbf{y})} \quad t > 0, \mathbf{x}, \mathbf{y} \in W_n^\circ, \tag{5.3}$$

where the entries in the determinant are solutions to the stochastic heat equation each driven by the same white noise.

From now on we denote by $\mathbf{x}_{k:l}$ the vector $(x_k, x_{k+1}, \dots, x_l)$ for $k < l$ and its Vandermonde determinant by $\Delta(\mathbf{x}_{k:l}) = \prod_{k \leq i < j \leq l} (x_i - x_j)$. As always we denote the vector $(1,\ldots,1)$ of any dimension by $\mathbf{1}$. In order to prove Theorem 5.1.2, we need the following intermediate step.

Proposition 5.1.3. *For $n \geq 2$ and $\mathbf{y} \in W_n^\circ$, let $\tau_n(\mathbf{y}) = \Delta(\mathbf{y})^{-1}\det[g_i(y_j)]_{i,j=1}^n$, $\tau_1(i,\cdot) = g_i(\cdot)$ and $\tau_0 \equiv 1$, where $g_i(y) : \mathbb{R} \to \mathbb{R}$ are continuous and strictly positive. For $k \leq n$ and $1 \leq a,b,c,d \leq n$, $a < b$, $c < d$, let $\tau_k\big((a:b), \mathbf{y}_{c:d}\big)$ be the ratio of $k \times k$ determinants $\Delta(\mathbf{y}_{c:d})^{-1}\det[g_i(y_j)]_{a\leq i\leq b; c\leq j\leq d}$. Moreover, assume that all the τ_k's for any (a,b,c,d) are continuous and strictly positive on W_k. Then the following identity holds for all $\mathbf{y} \in W_n^\circ$,*

$$\begin{aligned}
&\Delta(\mathbf{y})\tau_n(\mathbf{y}) \\
&= \prod_{i=1}^n g_n(y_i) \int_{\mathrm{GT}(\mathbf{y})} \prod_{k=1}^{n-1}\prod_{i=1}^{n-k} k \frac{\tau_{k-1}\big((n-k+2:n), z_i^{n-k}\mathbf{1}\big)\tau_{k+1}\big((n-k:n), z_i^{n-k}\mathbf{1}\big)}{\tau_k\big((n-k+1:n), z_i^{n-k}\mathbf{1}\big)^2}\, \mathrm{d}z_i^{n-k}.
\end{aligned} \tag{5.4}$$

In the sequel we will apply the above result with $g_i(\cdot) = u(t, x_i, \cdot)$, then we see from the determinantal expression (5.3) of M_n that the formula of the proposition is, up to a factor of the Vandermonde determinant, equation (5.2) with all the $\mathbf{x}$ coordinates being distinct. Theorem 5.1.2 then follows upon dividing both sides by $\Delta(\mathbf{x})$ and taking the limit as $\mathbf{x} \to x\mathbf{1}$ using the continuity of M_n. Thus, (5.4) can be viewed as a more general version of the integral formula (5.2).

Another consequence of the continuity of M_n on the whole of the Weyl chamber and its determinantal expression is that the ratio of two solutions to the stochastic heat equation with different delta initial data is differentiable with respect to y almost surely and its derivative can be expressed in terms of M_2 and u. Indeed, by properties of determinants we have

$$(x_1 - x_2)\frac{M_2(t,\mathbf{x},\mathbf{y})}{u(t,x_2,y_1)u(t,x_2,y_2)} = \frac{1}{y_1 - y_2}\left(\frac{u(t,x_1,y_1)}{u(t,x_2,y_1)} - \frac{u(t,x_1,y_2)}{u(t,x_2,y_2)}\right).$$

Let $y_2 = z$ and $y_1 = z + h$ for $h > 0$. Since $y \mapsto u(t,x,y)$ and $\mathbf{y} \mapsto M_2(t,\mathbf{x},\mathbf{y})$ are continuous almost surely, the above converges as $h \to 0$ to

$$\begin{aligned}
(x_1 - x_2)\frac{M_2(t,\mathbf{x},z\mathbf{1})}{u(t,x_2,z)^2} &= \lim_{h\to 0}\frac{1}{h}\left(\frac{u(t,x_1,z+h)}{u(t,x_2,z+h)} - \frac{u(t,x_1,z)}{u(t,x_2,z)}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{u(t,x_1,y)}{u(t,x_2,y)}\right)\bigg|_{y=z}.
\end{aligned} \tag{5.5}$$

It was shown in [Hai13] that the difference of two solutions to the KPZ equation (with the same white noise) starting from two different Hölder continuous initial data is in $C^{\frac{3}{2}-\varepsilon}$ for every $\varepsilon > 0$. Here we have shown that for delta initial data, the ratio of solutions to the SHE is differentiable and gave a formula for the derivative. Moreover, the same method shows that an identical formula but with the role of $\mathbf{x}$ and $\mathbf{y}$ reversed holds for the derivative with respect to x.

We now discuss the connection of $Z_n(t,x,y)$ with integrable systems. The proofs of some of the statements below can be found in [OW11, Section 3]. Suppose that we replace the space-time white noise with a smooth space-time potential in the definition of Z_n then it can be shown that Z_n is given by the bi-directional Wronskian

$$Z_n(t,x,y) = c_n t^{n(n-1)/2} \det[\partial_x^{i-1}\partial_y^{j-1} u(t,x,y)]_{i,j=1}^n,$$

where $u(t,x,y)$ is the solution to the heat equation (3.2) driven by the smooth potential. Now let $\tau_n = \det[\partial_x^{i-1}\partial_y^{j-1} u(t,x,y)]_{i,j=1}^n$ then τ_n satisfy the two-dimensional Toda equations (2DTE)

$$\partial_{xy} q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}, \quad n \geq 1,$$

where $q_n = \log(\tau_n/\tau_{n-1})$ or equivalently,

$$\partial_{xy} \log \tau_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}, \tag{5.6}$$

with the convention that $\tau_0 \equiv 1$. Evaluating the derivative and rearranging we obtain

$$\tau_n \partial_{xy}\tau_n - (\partial_x \tau_n)(\partial_y \tau_n) = \tau_{n-1}\tau_{n+1}. \tag{5.7}$$

We introduce the following notation. For an $(n+1)\times(n+1)$ determinant D, let $D\begin{bmatrix} i \\ j \end{bmatrix}$ be the $n \times n$ determinant obtained from D by removing the ith row and the jth column and similarly let $D\begin{bmatrix} i & j \\ k & l \end{bmatrix}$ be the $(n-1)\times(n-1)$ determinant obtained from D by removing the ith and jth rows and the kth and lth columns. Then, we have

$$\begin{aligned}
\tau_n &= \tau_{n+1}\begin{bmatrix} n+1 \\ n+1 \end{bmatrix}, \\
\tau_{n-1} &= \tau_{n+1}\begin{bmatrix} n & n+1 \\ n & n+1 \end{bmatrix}, \\
\partial_x \tau_n &= \tau_{n+1}\begin{bmatrix} n \\ n+1 \end{bmatrix}, \\
\partial_y \tau_n &= \tau_{n+1}\begin{bmatrix} n+1 \\ n \end{bmatrix}, \\
\partial_{xy} \tau_n &= \tau_{n+1}\begin{bmatrix} n \\ n \end{bmatrix},
\end{aligned}$$

where the last three expressions follows from the multi-linearity of determinants. Then, by properties of Wronskians, (5.7) can be written as

$$\tau_{n+1}\begin{bmatrix} n+1 \\ n+1 \end{bmatrix}\tau_{n+1}\begin{bmatrix} n \\ n \end{bmatrix} - \tau_{n+1}\begin{bmatrix} n \\ n+1 \end{bmatrix}\tau_{n+1}\begin{bmatrix} n+1 \\ n \end{bmatrix} = \tau_{n+1}\begin{bmatrix} n & n+1 \\ n & n+1 \end{bmatrix}\tau_{n+1},$$

which is nothing but the Jacobi identity for determinants, [Hir04, equation 2.73].

In fact, one can replace u with any smooth function of x and y in the definition of τ_n and it will still satisfy the 2DTE since, written in the form of (5.7), the equation is just the Jacobi identity. It was shown in [Hir04, Chapter 3.6] that $\det[\partial_x^{i-1}\partial_y^{j-1}\Psi(x,y)]_{i,j=1}^n$ is a solution to (5.7) where the function Ψ is chosen so that the boundary condition $\tau_{N+1} = \chi(x)\Phi(y)$ for some N is satisfied where χ and Φ are arbitrary functions in x and y respectively.

Using essentially the same argument as above, it was shown in [OW11, Lemma 3.6] that the following also holds for $k \geq 1$

$$\tau_n \partial_x^k \partial_y \tau_n - (\partial_x^k \tau_n)(\partial_y \tau_n) = \tau_{n-1}\partial_x^{k-1}\tau_{n+1},$$

and from this it follows that

$$\big(\partial_y(\partial_x^k \tau_n/\tau_n)\big)/T_n = (\partial_x^{k-1}\tau_{n+1})/\tau_{n+1},$$

where $T_n = \tau_{n-1}\tau_{n+1}/\tau_n^2$. Using this, the authors of [OW11] showed that the integral formula (5.2) holds in the case of a smooth space-time potential. We demonstrate this here in the case $n=2$. By the previous display, we have $\partial_y(\partial_x u/u) = T_1$ and so

$$\det\left[\frac{\partial_x^{i-1}u(t,x,y_j)}{u(t,x,y_j)}\right]_{i,j=1}^2 = \int_{y_2}^{y_1} \partial_y\left(\frac{\partial_x u(t,x,y)}{u(t,x,y)}\right) \mathrm{d}y = \int_{y_2}^{y_1} \frac{\tau_2(t,x,y)}{u(t,x,y)^2}\,\mathrm{d}y.$$

Combining this with the fact that $M_2(t, x\mathbf{1}, \mathbf{y}) = c_2 \Delta(\mathbf{y})^{-1}\det[\partial_x^{i-1}u(t,x,y_j)]_{i,j=1}^2$ completes the proof in the case $n=2$. In the white noise setting, none of the derivatives above exists but we do have that the ratio of two solutions to the SHE is differentiable, see (5.5), and more generally we have Lemma 5.2.1 which takes (5.5) as inspiration. Thus, we can apply a similar procedure to the one described above in order to prove Proposition 5.1.3. Note that in the smooth case we work with M_n evaluated at $\mathbf{x} = x\mathbf{1}$ but in the white noise setting, as we shall see below, we essentially start with $\mathbf{x} \in W_n$ (by considering distinct g_i's) so that Lemma 5.2.1 can be applied and at the end we take the limit as $\mathbf{x} \to x\mathbf{1}$.

The Jacobi identity comes into play again in the proof of Lemma 5.2.1 using the fact that τ_n is given by determinants. By a similar method, we also show that for any fixed time $t > 0$, $(\tilde{Z}_n(t,\cdot,\cdot) : n \geq 1)$ defined by $\tilde{Z}_n = c_n^{-1} Z_n$ satisfies an integrated form of the 2DTE. By this we mean that for any $x_1 > x_2$ and $y_1 > y_2$

$$\log\frac{\tilde{Z}_n(t,x_1,y_1)}{\tilde{Z}_n(t,x_1,y_2)} - \log\frac{\tilde{Z}_n(t,x_2,y_1)}{\tilde{Z}_n(t,x_2,y_2)} = t^{-n(n-1)/2}\int_{x_2}^{x_1}\int_{y_2}^{y_1} \frac{\tilde{Z}_{n-1}(t,x,y)\tilde{Z}_{n+1}(t,x,y)}{\tilde{Z}_n(t,x,y)^2}\,\mathrm{d}y\mathrm{d}x. \tag{5.8}$$

This suggests that one can interpret the fixed time Cole–Hopf solution to the KPZ equation with narrow wedge initial condition as the first element of the two-dimensional Toda chain.

The outline of the chapter is as follows. In Section 5.2 we shall prove Proposition 5.1.3 from which Theorem 5.1.2 follows easily and we show that Z_n satisfies an integrated form of the 2D Toda equations. Finally, in Section 5.3 we prove the Markov property of the multi-layer process (5.1) using Theorem 5.1.2 and the Markov property of M_n which is a natural consequence of it satisfying an evolution equation.

5.2 Proof of the Integral Formula

We first prove the following intermediate result which allows us to prove Proposition 5.1.3 "layer by layer". For $1 \leq a \leq b \leq c \leq n$ we use the notation $(a, b : c)$ to denote the sequence of indices $(a, b, b+1, \ldots, c)$. Then

$$\tau_k\big((a, b : c), \mathbf{y}_{1:k}\big) = \frac{\det[g_i(y_j)]_{i=a,b,b+1,\ldots,c;j=1,\ldots,k}}{\Delta(\mathbf{y}_{1:k})}.$$

If $b = c$, we set $(a, b : c) = (a, b)$ and if $b > c$ then $(a, b : c) = (a)$. Recall that all of the τ_k's are assumed to be continuous and strictly positive on the whole of W_k.

Lemma 5.2.1. *Let $n \geq 1$, suppose $g_1, \ldots, g_n$ satisfy the assumptions of Proposition 5.1.3. For all $1 \leq k < n$, $1 \leq i \leq n-k$ and $a < b$, the following holds*

$$\frac{\tau_k\big((i, n-k+2 : n), b\mathbf{1}\big)}{\tau_k\big((n-k+1 : n), b\mathbf{1}\big)} - \frac{\tau_k\big((i, n-k+2 : n), a\mathbf{1}\big)}{\tau_k\big((n-k+1 : n), a\mathbf{1}\big)}$$
$$= k \int_a^b \frac{\tau_{k-1}\big((n-k+2 : n), y\mathbf{1}\big)\tau_{k+1}\big((i, n-k+1 : n), y\mathbf{1}\big)}{\tau_k\big((n-k+1 : n), y\mathbf{1}\big)^2}\, \mathrm{d}y.$$

Proof. We first prove that for $\mathbf{y} \in W_{k+1}^{\circ}$

$$\frac{1}{y_1 - y_{k+1}}\left(\frac{\tau_k\big((i, n-k+2 : n), \mathbf{y}_{1:k}\big)}{\tau_k\big((n-k+1 : n), \mathbf{y}_{1:k}\big)} - \frac{\tau_k\big((i, n-k+2 : n), \mathbf{y}_{2:k+1}\big)}{\tau_k\big((n-k+1 : n), \mathbf{y}_{2:k+1}\big)}\right)$$
$$= \frac{\tau_{k-1}\big((n-k+2 : n), \mathbf{y}_{2:k}\big)\tau_{k+1}\big((i, n-k+1), \mathbf{y}_{1:k+1}\big)}{\tau_k\big((n-k+1 : n), \mathbf{y}_{1:k}\big)\tau_k\big((n-k+1 : n), \mathbf{y}_{2:k+1}\big)}. \tag{5.9}$$

Observe that

$$\frac{\Delta(\mathbf{y}_{1:k})\Delta(\mathbf{y}_{2:k+1})}{\Delta(\mathbf{y}_{2:k})\Delta(\mathbf{y}_{1:k+1})} = \frac{1}{y_1 - y_{k+1}}, \tag{5.10}$$

which can be seen by a direct calculation and noticing that there cannot be a factor of $(y_1 - y_{k+1})$ in the numerator on the left hand side. Define the $(k+1) \times (k+1)$ determinant $\tilde{\tau}_{k+1}$ by $\Delta(\mathbf{y}_{1:k+1})\tau_{k+1}\big((i, n-k+1 : n), \mathbf{y}_{1:k+1}\big)$ then by (5.10), in the notation introduced below equation (5.7), equation (5.9) is equivalent to

$$\tilde{\tau}_{k+1}\begin{bmatrix} 2 \\ k+1 \end{bmatrix}\tilde{\tau}_{k+1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \tilde{\tau}_{k+1}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\tilde{\tau}_{k+1}\begin{bmatrix} 1 \\ k+1 \end{bmatrix} = \tilde{\tau}_{k+1}\begin{bmatrix} 1 & 2 \\ 1 & k+1 \end{bmatrix}\tilde{\tau}_{k+1}. \tag{5.11}$$

Now let T be the $(k+1) \times (k+1)$ determinant obtained from $\tilde{\tau}_{k+1}$ by interchanging the second and $(k+1)$th column of $\tilde{\tau}_{k+1}$. Then by the properties of determinants

$$T\begin{bmatrix}2\\2\end{bmatrix} = (-1)^{k-2}\,\tilde{\tau}_{k+1}\begin{bmatrix}2\\k+1\end{bmatrix} \qquad T\begin{bmatrix}1\\1\end{bmatrix} = -\tilde{\tau}_{k+1}\begin{bmatrix}1\\1\end{bmatrix}$$

$$T\begin{bmatrix}1\\2\end{bmatrix} = (-1)^{k-2}\,\tilde{\tau}_{k+1}\begin{bmatrix}1\\k+1\end{bmatrix} \qquad T\begin{bmatrix}2\\1\end{bmatrix} = -\tilde{\tau}_{k+1}\begin{bmatrix}2\\1\end{bmatrix}$$

$$T\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix} = (-1)^{k-2}\,\tilde{\tau}_{k+1}\begin{bmatrix}1 & 2\\1 & k+1\end{bmatrix} \qquad T = -\tilde{\tau}_{k+1}$$

Therefore, equation (5.11) can be rewritten as

$$T\begin{bmatrix}2\\2\end{bmatrix} T\begin{bmatrix}1\\1\end{bmatrix} - T\begin{bmatrix}2\\1\end{bmatrix} T\begin{bmatrix}1\\2\end{bmatrix} = T\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix} T, \tag{5.12}$$

which is simply the Jacobi identity for determinants [Hir04, equation 2.73].

For $h > 0$ and $z \in \mathbb{R}$, let $\mathbf{y} = (y_1, y_2, \ldots, y_{k+1}) = (z + kh, z + (k-1)h, \ldots, z)$. Then integrating both sides of (5.9) with respect to z over the interval $[a, b]$, the left hand side becomes

$$\frac{1}{kh}\int_a^b f(z + kh, \ldots, z + h)\,\mathrm{d}z - \frac{1}{kh}\int_a^b f(z + (k-1)h, \ldots, z)\,\mathrm{d}z, \tag{5.13}$$

where

$$f(\cdot) = \frac{\tau_k\big((i, n-k+2 : n),\ \cdot\ \big)}{\tau_k\big((n-k+1 : n),\ \cdot\ \big)}.$$

Making the change of variables $z \mapsto z + h$ in the first integral, we can rewrite (5.13), for h small enough, as

$$\begin{aligned}\frac{1}{kh}&\int_{a+h}^{b+h} f(z + (k-1)h, \ldots, z)\,\mathrm{d}z - \frac{1}{kh}\int_a^b f(z + (k-1)h, \ldots, z)\,\mathrm{d}z\\ &= \frac{1}{kh}\int_b^{b+h} f(z + (k-1)h, \ldots, z)\,\mathrm{d}z - \frac{1}{kh}\int_a^{a+h} f(z + (k-1)h, \ldots, z)\,\mathrm{d}z,\end{aligned}$$

which converges as $h \to 0$, by the continuity of f, to

$$\frac{1}{k}\big(f(b, \ldots, b) - f(a, \ldots, a)\big).$$

Indeed, for all $\varepsilon > 0$, there exists $h_0 > 0$ such that if $h < h_0$ then

$$|f(z + (k-1)h, \ldots, z) - f(b, \ldots, b)| < \varepsilon, \quad z \in [b, b+h].$$

Then for $h < h_0$,

$$\begin{aligned}\left|\frac{1}{h}\int_b^{b+h} f(z+(k-1)h,\dots,z)\,\mathrm{d}z - f(b,\dots,b)\right|&\\ &\le \frac{1}{h}\int_b^{b+h} |f(z+(k-1)h,\dots,z) - f(b,\dots,b)|\,\mathrm{d}z\\ &\le \varepsilon\frac{1}{h}\int_b^{b+h} 1\,\mathrm{d}z = \varepsilon.\end{aligned}$$

The other term can be shown to converge to $f(a,\dots,a)$ in the same manner.

On the other hand, the integral of the right hand side of (5.9) converges as $h \to 0$ to

$$\int_a^b \frac{\tau_{k-1}\big((n-k+2:n),z\mathbf{1}\big)\tau_{k+1}\big((i,n-k+1:n),z\mathbf{1}\big)}{\tau_k\big((n-k+1:n),z\mathbf{1}\big)^2}\,\mathrm{d}z,$$

by the continuity of $\tau_{k-1}\tau_{k+1}/\tau_k^2$ and the dominated convergence theorem. The proof is now complete. $\square$

In the sequel we need the following result which is sometimes called the generalised Cauchy–Binet formula [Joh06, Proposition 2.10]. Let $(X,\mathscr{B},\mu)$ be a measure space. For measurable functions ϕ_i, ψ_i, $1 \le i \le n$ such that $\phi_i\psi_j$ is integrable for any i, j, we have

$$\det\left[\int_X \phi_i(x)\psi_j(x)\,\mathrm{d}\mu(x)\right]_{i,j=1}^n = \frac{1}{n!}\int_{X^n}\det[\phi_i(x_j)]_{i,j=1}^n\det[\psi_i(x_j)]_{i,j=1}^n\prod_{j=1}^n\mathrm{d}\mu(x_j).$$

Proof of Proposition 5.1.3. Fix $\mathbf{y} = (y_1,\dots,y_n) \in W_n^\circ$. For brevity we write $g_{ij} = g_i(y_j)$, then

$$\begin{aligned}\prod_{j=1}^n g_{nj}^{-1}\Delta(\mathbf{y})\tau_n\big((1:n),\mathbf{y}\big) &= \det\left[\frac{g_{ij}}{g_{nj}}\right]_{i,j=1}^n\\ &= \det\left[\frac{g_{ij}}{g_{nj}} - \frac{g_{ij+1}}{g_{nj+1}}\right]_{i,j=1}^{n-1}\\ &= \det\left[\int_{y_{j+1}}^{y_j}\frac{\tau_2\big((i,n),z\mathbf{1}\big)}{g_n(z)^2}\,\mathrm{d}z\right]_{i,j=1}^{n-1},\end{aligned}$$

where we have used Lemma 5.2.1 with $k=1$ in the last equality. Using the formula $1_{\mathbf{z}\prec\mathbf{y}} = \det[1_{y_j \ge z_i > y_{j+1}}]_{i,j=1}^{n-1}$ and the generalised Cauchy–Binet formula, the last line of the previous display is equal to

$$
\begin{aligned}
&\det\left[\int_{\mathbb{R}} 1_{y_j \geq z > y_{j+1}} \frac{\tau_2\big((i,n), z\mathbf{1}\big)}{g_n(z)^2}\, \mathrm{d}z\right]_{i,j=1}^{n-1} \\
&\quad = \int_{W_{n-1}} \det[1_{y_j \geq z_i^{n-1} > y_{j+1}}]_{i,j=1}^{n-1} \det\left[\frac{\tau_2\big((i,n), z_j^{n-1}\mathbf{1}\big)}{g_n(z_j^{n-1})^2}\right]_{i,j=1}^{n-1} \mathrm{d}\mathbf{z}^{n-1} \\
&\quad = \int_{\mathbf{z}^{n-1}\prec\mathbf{y}} \det\left[\frac{\tau_2\big((i,n), z_j^{n-1}\mathbf{1}\big)}{g_n(z_j^{n-1})^2}\right]_{i,j=1}^{n-1} \mathrm{d}\mathbf{z}^{n-1} \\
&\quad = \int_{\mathbf{z}^{n-1}\prec\mathbf{y}} \prod_{j=1}^{n-1} \frac{\tau_2\big((n-1,n), z_j^{n-1}\mathbf{1}\big)}{g_n(z_j^{n-1})^2} \det\left[\frac{\tau_2\big((i,n), z_j^{n-1}\mathbf{1}\big)}{\tau_2\big((n-1,n), z_j^{n-1}\mathbf{1}\big)}\right]_{i,j=1}^{n-1} \mathrm{d}\mathbf{z}^{n-1} \\
&\quad = \int_{\mathbf{z}^{n-1}\prec\mathbf{y}} \prod_{j=1}^{n-1} \frac{\tau_2\big((n-1,n), z_j^{n-1}\mathbf{1}\big)}{g_n(z_j^{n-1})^2} \\
&\qquad\quad \times \det\left[\frac{\tau_2\big((i,n), z_j^{n-1}\mathbf{1}\big)}{\tau_2\big((n-1,n), z_j^{n-1}\mathbf{1}\big)} - \frac{\tau_2\big((i,n), z_{j+1}^{n-1}\mathbf{1}\big)}{\tau_2\big((n-1,n), z_{j+1}^{n-1}\mathbf{1}\big)}\right]_{i,j=1}^{n-2} \mathrm{d}\mathbf{z}^{n-1}.
\end{aligned}
$$

Proceeding in the same manner and using Lemma 5.2.1 repeatedly we arrive at

$$
\begin{aligned}
&\int_{\mathbf{z}^{n-1}\prec\mathbf{y}} \cdots \int_{\mathbf{z}^2\prec\mathbf{z}^3} \prod_{k=1}^{n-2}\prod_{j=1}^{n-k} k \frac{\tau_{k-1}\big((n-k+2:n), z_j^{n-k}\mathbf{1}\big)\tau_{k+1}\big((n-k:n), z_j^{n-k}\mathbf{1}\big)}{\tau_k\big((n-k+1:n), z_j^{n-k}\mathbf{1}\big)^2} \\
&\quad \times \det\left[\frac{\tau_{n-1}\big((i,3:n), z_j^2\mathbf{1}\big)}{\tau_{n-1}\big((2:n), z_j^2\mathbf{1}\big)}\right]_{i,j=1}^{2} \mathrm{d}z_j^{n-k} \\
&= \int_{\mathbf{z}^{n-1}\prec\mathbf{y}} \cdots \int_{\mathbf{z}^2\prec\mathbf{z}^3} \prod_{k=1}^{n-2}\prod_{j=1}^{n-k} k \frac{\tau_{k-1}\big((n-k+2:n), z_j^{n-k}\mathbf{1}\big)\tau_{k+1}\big((n-k:n), z_j^{n-k}\mathbf{1}\big)}{\tau_k\big((n-k+1:n), z_j^{n-k}\mathbf{1}\big)^2} \\
&\quad \times \int_{z_2^2}^{z_1^2} (n-1) \frac{\tau_{n-2}\big((3:n), z_1^1\mathbf{1}\big)\tau_n\big((1:n), z_1^1\mathbf{1}\big)}{\tau_{n-1}\big((2:n), z_1^1\mathbf{1}\big)^2} \,\mathrm{d}z_1^1 \mathrm{d}z_i^{n-k} \\
&= \int_{\mathrm{GT}(\mathbf{y})} \prod_{k=1}^{n-1}\prod_{j=1}^{n-k} k \frac{\tau_{k-1}\big((n-k+2:n), z_j^{n-k}\mathbf{1}\big)\tau_{k+1}\big((n-k:n), z_j^{n-k}\mathbf{1}\big)}{\tau_k\big((n-k+1:n), z_j^{n-k}\mathbf{1}\big)^2} \,\mathrm{d}z_i^{n-k}.
\end{aligned}
$$

□

Theorem 5.1.2 now follows from Proposition 5.1.3 in a straightforward manner.

Proof of Theorem 5.1.2. Theorem 3.1.2 and 4.1.1 shows that for all $n \geq 1$, $M_n(t, \mathbf{x}, \mathbf{y})$ is a strictly positive, continuous function on $(0, \infty) \times W_n \times W_n$ almost surely and by (5.3), $M_n(t, \mathbf{x}, \mathbf{y}) = \det[u(t, x_i, y_j)]_{i.j=1}^n / \Delta(\mathbf{x})\Delta(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in W_n^\circ$. Fix $t > 0$ then an application of Proposition 5.1.3 with $g_i(y_j) = u(t, x_i, y_j)$, $\tau_n = \tilde{M}_n := \det[u(t, x_i, y_j)]/\Delta(\mathbf{y})$, with $\tilde{M}_0 \equiv 1$,

$\tilde{M}_1 = u$, shows that for all $\mathbf{x}$, $\mathbf{y} \in W_n^\circ$

$$\begin{aligned}&\Delta(\mathbf{y})\tilde{M}_n(t,\mathbf{x},\mathbf{y})\\ &= \prod_{i=1}^n u(t,x_n,y_i) \int_{\mathrm{GT}(\mathbf{y})} \prod_{k=1}^{n-1}\prod_{i=1}^{n-k} k \frac{\tilde{M}_{k-1}(t,\mathbf{x}_{n-k+2:n}, z_i^{n-k}\mathbf{1})\tilde{M}_{k+1}(t,\mathbf{x}_{n-k:n}, z_i^{n-k}\mathbf{1})}{\tilde{M}_k(t,\mathbf{x}_{n-k+1:n}, z_i^{n-k}\mathbf{1})^2}\,\mathrm{d}z_i^{n-k}.\end{aligned} \tag{5.14}$$

Observe that

$$\begin{aligned}&\frac{\tilde{M}_{k-1}(t,\mathbf{x}_{n-k+2:n}, z_i^{n-k}\mathbf{1})\tilde{M}_{k+1}(t,\mathbf{x}_{n-k:n}, z_i^{n-k}\mathbf{1})}{\tilde{M}_k(t,\mathbf{x}_{n-k+1:n}, z_i^{n-k}\mathbf{1})^2}\\ &\qquad = V_k(\mathbf{x})\frac{M_{k-1}(t,\mathbf{x}_{n-k+2:n}, z_i^{n-k}\mathbf{1})M_{k+1}(t,\mathbf{x}_{n-k:n}, z_i^{n-k}\mathbf{1})}{M_k(t,\mathbf{x}_{n-k+1:n}, z_i^{n-k}\mathbf{1})^2}\end{aligned}$$

where

$$V_k(\mathbf{x}) = \begin{cases} \frac{\prod_{i=n-k+1}^n (x_{n-k}-x_i)}{\prod_{i=n-k+2}^n (x_{n-k+1}-x_i)} & k \geq 2, \\ x_{n-1}-x_n & k=1. \end{cases}$$

Therefore,

$$\begin{aligned}&\prod_{k=1}^{n-1}\prod_{i=1}^{n-k} V_k(\mathbf{x})\\ &\qquad = \frac{\prod_{i=2}^n (x_1-x_i)}{\prod_{i=3}^n (x_2-x_i)}\frac{\prod_{i=3}^n (x_2-x_i)^2}{\prod_{i=4}^n (x_3-x_i)^2}\cdots\frac{\prod_{i=n-1}^n (x_{n-2}-x_i)^{n-2}}{(x_{n-1}-x_n)^{n-2}}(x_{n-1}-x_n)^{n-1}\\ &\qquad = \prod_{i=2}^n (x_1-x_i)\prod_{i=3}^n (x_2-x_i)\cdots\prod_{i=n-1}^n (x_{n-2}-x_i)(x_{n-1}-x_n)\\ &\qquad = \Delta(\mathbf{x}).\end{aligned}$$

Thus, dividing equation (5.14) through by $\Delta(\mathbf{x})$ we obtain

$$\begin{aligned}&\Delta(\mathbf{y})M_n(t,\mathbf{x},\mathbf{y})\\ &= \prod_{i=1}^n u(t,x_n,y_i) \int_{\mathrm{GT}(\mathbf{y})} \prod_{k=1}^{n-1}\prod_{i=1}^{n-k} k \frac{M_{k-1}(t,\mathbf{x}_{n-k+2:n}, z_i^{n-k}\mathbf{1})M_{k+1}(t,\mathbf{x}_{n-k:n}, z_i^{n-k}\mathbf{1})}{M_k(t,\mathbf{x}_{n-k+1:n}, z_i^{n-k}\mathbf{1})^2}\,\mathrm{d}z_i^{n-k}.\end{aligned} \tag{5.15}$$

Recall that $M_n(t,x\mathbf{1},y\mathbf{1}) = c_{n,t}Z_n(t,x,y)$ and so taking limits as $\mathbf{x} \to x\mathbf{1}$, the left hand side of (5.15) converges to $\Delta(\mathbf{y})M_n(t,x\mathbf{1},\mathbf{y})$ by the continuity of M_n. By the continuity of u and $M_{k-1}M_{k+1}/M_k^2$ and by the dominated convergence theorem, the right hand side of (5.15), noting that $c_{k-1,t}c_{k+1,t}/c_{k,t}^2 = 1/kt$, converges to

$$\prod_{i=1}^n u(t,x,y_i)\int_{\mathrm{GT}(\mathbf{y})}\prod_{k=1}^{n-1}\prod_{i=1}^{n-k}\frac{1}{t}\frac{Z_{k-1}(t,x,z_i^{n-k})Z_{k+1}(t,x,z_i^{n-k})}{Z_k(t,x,z_i^{n-k})^2}\,\mathrm{d}z_i^{n-k}.$$

Rearranging gives formula (5.2) and hence completes the proof. $\square$

We now prove that Z_n satisfies an integrated form of the 2D Toda equations, see (5.8). The proof uses again the Jacobi identity for determinants and the continuity of M_n. Let $g(x,y):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous and strictly positive and for $\mathbf{x}$, $\mathbf{y}\in W_n$ let $\tau_n(\mathbf{x},\mathbf{y})=\frac{\det[g(x_i,y_j)]_{i,j=1}^n}{\Delta(\mathbf{x})\Delta(\mathbf{y})}$. Assume that τ_n is continuous and strictly positive on W_n for all n. Denote $\tilde{\tau}_n=c_n^{-2}\tau_n$, then we have the following

Lemma 5.2.2. *For any $n\geq 1$ and $x_1>x_2$, $y_1>y_2$ we have*

$$\log\frac{\tilde{\tau}_n(x_1\mathbf{1},y_1\mathbf{1})}{\tilde{\tau}_n(x_1\mathbf{1},y_2\mathbf{1})}-\log\frac{\tilde{\tau}_n(x_2\mathbf{1},y_1\mathbf{1})}{\tilde{\tau}_n(x_2\mathbf{1},y_2\mathbf{1})}=\int_{x_2}^{x_1}\int_{y_2}^{y_1}\frac{\tilde{\tau}_{n-1}(x\mathbf{1},y\mathbf{1})\tilde{\tau}_{n+1}(x\mathbf{1},y\mathbf{1})}{\tilde{\tau}_n(x\mathbf{1},y\mathbf{1})^2}\,\mathrm{d}y\mathrm{d}x.$$

Proof. Using the Jacobi identity and by a direct calculation involving product and ratio of Vandermonde determinants, we have

$$\begin{aligned}\frac{1}{y_1-y_{n+1}}&\left(\frac{\tau_n\big(\mathbf{x}_{1:n},\mathbf{y}_{1:n}\big)}{\tau_n\big(\mathbf{x}_{2:n+1},\mathbf{y}_{1:n}\big)}-\frac{\tau_n\big(\mathbf{x}_{1:n},\mathbf{y}_{2:n+1}\big)}{\tau_n\big(\mathbf{x}_{2:n+1}\mathbf{y}_{2:n+1}\big)}\right)\\&=(x_1-x_{n+1})\frac{\tau_{n-1}\big(\mathbf{x}_{2:n},\mathbf{y}_{2:n}\big)\tau_{n+1}\big(\mathbf{x}_{1:n+1},\mathbf{y}_{1:n+1}\big)}{\tau_n\big(\mathbf{x}_{2:n+1},\mathbf{y}_{1:n}\big)\tau_n\big(\mathbf{x}_{2:n+1},\mathbf{y}_{2:n+1}\big)}.\end{aligned}$$

The above is essentially equation (5.12) with $n+1$ in place of 2. From this it follows in the say way as in the proof of Lemma 5.2.1 that for any $a<b$

$$\begin{aligned}\frac{1}{x_1-x_{n+1}}&\left(\frac{\tau_n\big(\mathbf{x}_{1:n},b\mathbf{1}\big)}{\tau_n\big(\mathbf{x}_{2:n+1},b\mathbf{1}\big)}-\frac{\tau_n\big(\mathbf{x}_{1:n},a\mathbf{1}\big)}{\tau_n\big(\mathbf{x}_{2:n+1},a\mathbf{1}\big)}\right)\\&=\int_a^b n\frac{\tau_{n-1}\big(\mathbf{x}_{2:n},y\mathbf{1}\big)\tau_{n+1}\big(\mathbf{x}_{1:n+1},y\mathbf{1}\big)}{\tau_n\big(\mathbf{x}_{2:n+1},y\mathbf{1}\big)^2}\,\mathrm{d}y,\end{aligned}$$

which we can rearrange to obtain

$$\begin{aligned}\frac{1}{x_1-x_{n+1}}&\left(\frac{\tau_n\big(\mathbf{x}_{1:n},b\mathbf{1}\big)}{\tau_n\big(\mathbf{x}_{1:n},a\mathbf{1}\big)}-\frac{\tau_n\big(\mathbf{x}_{2:n+1},b\mathbf{1}\big)}{\tau_n\big(\mathbf{x}_{2:n+1},a\mathbf{1}\big)}\right)\\&=\frac{\tau_n\big(\mathbf{x}_{2:n+1},b\mathbf{1}\big)}{\tau_n\big(\mathbf{x}_{1:n},a\mathbf{1}\big)}\int_a^b n\frac{\tau_{n-1}\big(\mathbf{x}_{2:n},y\mathbf{1}\big)\tau_{n+1}\big(\mathbf{x}_{1:n+1},y\mathbf{1}\big)}{\tau_n\big(\mathbf{x}_{2:n+1},y\mathbf{1}\big)^2}\,\mathrm{d}y.\end{aligned}$$

Let $x_{n+1}=x$, $x_n=x+h,\ldots,x_1=x+nh$ and integrate both sides of the above with respect to x over the interval $[c,d]$. In the same way as in the proof of Lemma 5.2.1, the left hand side is of the form

$$\frac{1}{nh}\int_d^{d+h}f(x+(k-1)h,\ldots,x)\,\mathrm{d}x-\frac{1}{nh}\int_c^{c+h}f(x+(k-1)h,\ldots,x)\,\mathrm{d}x,$$

which converges as $h\to 0$ to $\frac{1}{n}\big(f(d,\ldots,d)-f(c,\ldots,c)\big)$, where $f(\cdot)=\frac{\tau_n(\cdot,b\mathbf{1})}{\tau_n(\cdot,a\mathbf{1})}$. On the other hand, the right hand side by the continuity of τ_n converges to

$$\int_c^d \frac{\tau_n\big(x\mathbf{1}, b\mathbf{1}\big)}{\tau_n\big(x\mathbf{1}, a\mathbf{1}\big)} \int_a^b n \frac{\tau_{n-1}\big(x\mathbf{1}, y\mathbf{1}\big)\tau_{n+1}\big(x\mathbf{1}, y\mathbf{1}\big)}{\tau_n\big(x\mathbf{1}, y\mathbf{1}\big)^2} \,\mathrm{d}y\mathrm{d}x.$$

By the continuity of the integrand and the fundamental theorem of calculus, this implies that

$$\frac{\tau_n(x\mathbf{1}, a\mathbf{1})}{\tau_n(x\mathbf{1}, b\mathbf{1})} \frac{\partial}{\partial x}\left(\frac{\tau_n(x\mathbf{1}, b\mathbf{1})}{\tau_n(x\mathbf{1}, a\mathbf{1})}\right) = \int_a^b n^2 \frac{\tau_{n-1}(x\mathbf{1}, y\mathbf{1})\tau_{n+1}(x\mathbf{1}, y\mathbf{1})}{\tau_n(x\mathbf{1}, y\mathbf{1})^2} \,\mathrm{d}y.$$

Finally, integrating with respect to x and noting that $c_{n-1}c_{n+1}/c_n^2 = n^{-1}$ gives the desired result. $\square$

Fix $t > 0$, then applying the above lemma with $\tau_n(\mathbf{x}, \mathbf{y}) = M_n(t, \mathbf{x}, \mathbf{y})$ and recalling that $M_n(t, x\mathbf{1}, y\mathbf{1}) = c_{n,t} Z_n(t, x, y)$ gives equation (5.8) with $\tilde{Z}_n = c_n^{-1} Z_n$.

5.3 Proof of the Markov Property

We now prove the Markov property of the multi-layer process (5.1). We fix a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ as in Chapter 3. We will need the following, see also [OW11, Corollary 6.2].

Lemma 5.3.1. *For each $\mathbf{x} \in W_n$, $\big(M_n(t, \mathbf{x}, \cdot), t \geq 0\big)$ is a Markov process with respect to $(\mathscr{F}_t)_{t\geq 0}$ with state space $C\big(W_n, (0, \infty)\big)$.*

The fact that M_n is Markov follows from the following *flow property*: for all $\mathbf{x}$, $\mathbf{y} \in W_n$ and s, $t \geq 0$

$$M_n(s + t, \mathbf{x}, \mathbf{y}) = \int_{W_n} M_n(s, \mathbf{x}, \mathbf{z}) M_n^s(t, \mathbf{z}, \mathbf{y}) \Delta(\mathbf{z})^2 \,\mathrm{d}\mathbf{z},$$

almost surely, where M_n^s is defined by the chaos expansion (3.45) but with the shifted white noise $\dot{W}^s(\cdot, \cdot) := \dot{W}(s + \cdot, \cdot)$. The flow property is a consequence of (5.3), the generalised Cauchy–Binet formula and the corresponding flow property of the solution to the one-dimensional SHE. The Markov property follows since M_n^s is independent of $\mathscr{F}_s$. The Markov property is also natural given the fact that it satisfies an evolution equation. Indeed, using the Chapman–Kolmogorov equation for Q_t, one can show that

$$\begin{aligned} M_n(s + t, \mathbf{x}, \mathbf{y}) = \frac{1}{n!} &\int_{\mathbb{R}^n} M_n(s, \mathbf{x}, \mathbf{y}') Q_t(\mathbf{y}, \mathbf{y}') \,\mathrm{d}\mathbf{y}' \\ &+ A_n \int_0^t \int_{\mathbb{R}^n} Q_{t-u}(\mathbf{y}, \mathbf{y}') M_n(s + u, \mathbf{x}, \mathbf{y}') \,\mathrm{d}\mathbf{y}'_* \, W^s(\mathrm{d}u, \mathrm{d}y'_1), \end{aligned}$$

almost surely for all s, $t \geq 0$ and $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$. The Markov property then follows from a similar argument as for SDEs.

We are now ready to prove the main result of this chapter.

Proof of Theorem 5.1.1. Let $0 \le s < t$ and fix $x \in \mathbb{R}$. Let $C \in \mathscr{B}(\mathcal{C})$ be a Borel set where $\mathcal{C} := C(\mathbb{R}) \times \cdots \times C(\mathbb{R})$ and denote $\mathbf{Z}_n(t,x) := \big(Z_1(t,x,\cdot), \ldots, Z_n(t,x,\cdot)\big)$. It suffices to show that the conditional probability that $\mathbf{Z}_n(t,x) \in C$ given $\mathscr{F}_s$ is measurable with respect to $\sigma\big(\mathbf{Z}_n(s,x)\big)$ since then we have

$$\begin{aligned}\mathbb{P}\big[\mathbf{Z}_n(t,x) \in C | \sigma\big(\mathbf{Z}_n(s,x)\big)\big] &= \mathbb{E}\big[\mathbb{E}[1\{\mathbf{Z}_n(t,x) \in C\} | \mathscr{F}_s] \big| \sigma\big(\mathbf{Z}_n(s,x)\big)\big] \\ &= \mathbb{P}[\mathbf{Z}_n(t,x) \in C | \mathscr{F}_s] \quad \text{a.s.}\end{aligned}$$

By (3.10), $\mathbf{Z}_n(t,x)$ is proportional to $\big(M_1(t, x\mathbf{1}, \cdot\mathbf{1}), \ldots, M_n(t, x\mathbf{1}, \cdot\mathbf{1})\big)$ and since M_n is a Markov process, the conditional expectation given $\mathscr{F}_s$ of the latter is measurable with respect to $\big(M_1(s, x\mathbf{1}, \cdot), \ldots, M_n(s, x\mathbf{1}, \cdot)\big)$. However, by Theorem 5.1.2, for each $n \ge 1$, $M_n(s, x\mathbf{1}, \cdot)$ is a function of $\mathbf{Z}_n(s,x)$ and the result follows. $\square$

Appendix A

The Walsh Integral

We recall the basic properties of the Walsh stochastic integrals which appears throughout this thesis. For more details see the references [Wal86], [Kho09], [Kho14], [DQS11] and [Dal99].

Formally, a space-time white noise $\dot{W}$ is a distribution valued Gaussian random field with mean zero and covariance

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y),$$

where δ denotes the delta function at 0. The rigorous definition of white noise is the following

Definition A.0.1. *Let $\mathscr{B}_b(\mathbb{R}^d)$ be the collection of Borel measurable subsets of $\mathbb{R}^d$ with finite Lebesgue measure. A* white noise *on $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is a mean zero Gaussian random field $\{\dot{W}(A)\}_{A\in\mathscr{B}_b(\mathbb{R}^d)}$ with covariance function*

$$\mathbb{E}[\dot{W}(A)\dot{W}(B)] = |A \cap B|, \quad \textit{for all } A, B \in \mathscr{B}_b(\mathbb{R}^d),$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^d$.

One can show that the covariance function $(A \times B) \mapsto |A \cap B|$ is positive definite and so by the general theory of Gaussian processes, white noise in the above definition does exists.

It is convenient to "break off" one of the dimensions of $\mathbb{R}^d$ to play the role of time. We will only consider the *space-time white noise* $\big(W_t(A), t \geq 0, A \in \mathscr{B}_b(\mathbb{R})\big)$ defined by $W_t(A) := \dot{W}([0,t] \times A)$ defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$ such that W is $\mathscr{F}_t$-adapted and $W_t(A) - W_s(A)$ is independent of $\mathscr{F}_s$ for all $A \in \mathscr{B}_b(\mathbb{R})$. Then $W_t(A)$ is a worthy martingale measure [Kho09, Definition 5.20] and for a suitable random field $f(s,y)$ one can define the Walsh integral

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(s,y)\, W(\mathrm{d}s, \mathrm{d}y) \tag{A.1}$$

with respect to the martingale measure $\{W_t(A)\}$.

We say a random field f is elementary if it is of the form

$$f(s,y) = X1_{(a,b)}(t)1_A(y),$$

where $0 \leq a < b$, $A \in \mathscr{B}_b(\mathbb{R})$ and X is a $\mathscr{F}_a$ measurable random variable. A simple process is a finite linear combination of elementary random fields. The set of simple processes generates a σ-algebra $\mathscr{P}$ on $\mathbb{R}_+ \times \mathbb{R} \times \Omega$ called the predictable σ-algebra. We say a random field f is predictable if it is $\mathscr{P}$-measurable and that $f \in \mathscr{P}_2$ if it is predictable and

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \mathbb{E}[f(s,y)^2] \, \mathrm{d}y\mathrm{d}s < \infty. \tag{A.2}$$

According to Walsh [Wal86] the stochastic integral (A.1) is defined for all random fields $f \in \mathscr{P}_2$. The resulting integrals have the following isometry property:

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+} \int_{\mathbb{R}} f(s,y) \, W(\mathrm{d}s,\mathrm{d}y)\right)^2\right] = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \mathbb{E}[f(s,y)^2] \, \mathrm{d}y\mathrm{d}s < \infty. \tag{A.3}$$

The stochastic integral (A.1) is itself a worthy martingale measure. We also have the following useful inequality.

Proposition A.0.2 (Burkholder–Davis–Gundy)**.** *For all $p \geq 2$ there exists $c_p > 0$ such that*

$$\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}} f(s,y) \, W(\mathrm{d}s,\mathrm{d}y)\right|^p\right] \leq c_p^p \, \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}} f(s,y)^2 \, \mathrm{d}y\mathrm{d}s\right|^{p/2}\right]. \tag{A.4}$$

Moreover, for $p > 2$ Carlen and Krée have shown in [CK91] that $c_p \leq 2\sqrt{p}$ and that this is the optimal bound.

The following stochastic Fubini's theorem (see for example [Wal86, p.297] and [Kho09, Theorem 5.30]) will also be useful.

Proposition A.0.3. *Let $(A, \mathscr{A}, \mu)$ be a measure space and $f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \times A \to \mathbb{R}$ such that*

$$\int_{[0,t]\times\mathbb{R}\times\Omega\times A} f(s,y,u)^2 \, \mathrm{d}y \, \mathrm{d}s \, \mu(\mathrm{d}u) \, \mathrm{d}\mathbb{P} < \infty.$$

Then almost surely

$$\int_A \left(\int_0^t \int_{\mathbb{R}} f(s,y,u) \, W(\mathrm{d}s,\mathrm{d}y)\right)\mu(\mathrm{d}u) = \int_0^t \int_{\mathbb{R}} \left(\int_A f(s,y,u) \, \mu(\mathrm{d}u)\right) W(\mathrm{d}s,\mathrm{d}y).$$

We now turn our attention to multiple stochastic integrals which appear in the chaos series in the introduction. Let $k > 1$ and let $f \in L^2_S([0,t]^k \times \mathbb{R}^k)$ such that $f(\pi\mathbf{s}, \pi\mathbf{y}) = f(\mathbf{s},\mathbf{y})$ for all $(\mathbf{s},\mathbf{y}) \in [0,t]^k \times \mathbb{R}^k$ and $\pi \in S_k$ where S_k is the set of permutations of $\{1,\ldots,k\}$ and $\pi\mathbf{s} = (s_{\pi 1},\ldots,s_{\pi k})$. Let $A_1,\ldots,A_k$ be disjoint subsets of $[0,t]\times\mathbb{R}$. An elementary function in $L^2_S([0,t]^k \times \mathbb{R}^k)$ is a function of the form

$$f(\mathbf{s}, \mathbf{y}) = \sum_{\pi \in S_k} \prod_{i=1}^{k} 1\{(s_{\pi i}, y_{\pi i}) \in A_i\}. \tag{A.5}$$

For such f we define the k-fold integral by

$$(f \cdot W)_k(t) = \int_{[0,t]^k} \int_{\mathbb{R}^k} f(\mathbf{s}, \mathbf{y}) \, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) = k! \prod_{i=1}^{k} \dot{W}(A_i).$$

It can be shown that linear combinations of functions of the form (A.5) are dense in $L^2_S([0,t]^k \times \mathbb{R}^k)$ and that for an elementary f, the integral $(f \cdot W)_k$ satisfies an Itô isometry, hence for a general $f \in L^2_S([0,t]^k \times \mathbb{R}^k)$, we define $(f \cdot W)_k = \lim_{n\to\infty}(f_n \cdot W)_k$ where $\{f_n\}_{n\geq 1}$ is a sequence of elementary functions such that $f_n \to f$ in $L^2([0,t]^k \times \mathbb{R}^k)$. The resulting integral is a mean zero random variable with covariance given by

$$\mathbb{E}[(f \cdot W)_k(t)(g \cdot W)_k(t)] = (f, g)_{L^2([0,t]^k \times \mathbb{R}^k)}. \tag{A.6}$$

For $f \in L^2([0,t]^k \times \mathbb{R}^k)$ that are not symmetric, we define its integral by first symmetrising f via

$$\tilde{f}(\mathbf{s}, \mathbf{y}) := \frac{1}{k!} \sum_{\pi \in S_k} f(\pi \mathbf{s}, \pi \mathbf{y}),$$

and then define

$$(f \cdot W)_k(t) = (\tilde{f} \cdot W)_k(t).$$

Let $\Delta_k(t) = \{0 < s_1 < s_2 < \cdots < s_k < t\}$ then for functions f defined on $\Delta_k(t) \times \mathbb{R}^k$, for example the k-point correlation function R_k appearing in (1.15), we first extend it to a function on $[0,t]^k$ by setting it to be zero for $\mathbf{s} \notin \Delta_k(t)$ and then define

$$\int_{\Delta_k(t)} \int_{\mathbb{R}^k} f(\mathbf{s}, \mathbf{y}) \, W^{\otimes k}(\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{y}) := (\tilde{f} \cdot W)_k(t).$$

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